Math 30-2: U2L3 Teacher Notes

Permutations When All Objects are Distinguishable

Key Math Learnings:

By the end of this lesson, you will learn the following concepts:

4.1 Represent and solve counting problems, using a graphic organizer.

4.4 Solve a contextual counting problem, using the fundamental counting principle, and explain the reasoning.

5.2 Determine, with or without technology, the value of a factorial.

5.3 Simplify a numeric or an algebraic fraction that contains factorials in both the numerator and denominator.

5.4 Solve an equation that involves factorials.

5.5 Determine the number of permutations of n elements taken r at a time.

5.8 Generalize strategies for determining the number of permutations of n elements taken r at a time.

Permutation Notation

Let's look at an example:

How many different seven digit numbers can be made from the digits 1 through 7 with no repetition of numbers allowed (ie. can't use the digit "5" twice)?

Solution:

Let the number be represented by seven blanks $\_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_$

The first digit could be any of the seven digits, so place a 7 in the blank $\underline{7}\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_$

The second digit could be any one of the six remaining digits, since no digit can be repeated, so place a 6 in the blank $\underline{7}\ \underline{6}\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_$

The third could be any one of the five remaining digits and so on.....

$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7! = 5040$

Therefore 5040 numbers can be made from these seven digits.

In the example above, when you are asked to find how many different seven digit numbers can be made from the numbers 1 to 7, you used factorial notation to find the answer: 7!

This example represents a permutation (an arrangement) of seven digits taken seven at a time (meaning all 7 objects are used up).

n! can be written as ${}_nP_n$ and read as “n permute n”, which means that all n objects are included in each arrangement. ${}_nP_n$ is called **permutation notation** (as is ${}_nP_r$ which we will study next)

The Number of Permutations of *n* different objects taken all at a time is *n*!

$${}_nP_n = n!$$

For Example: ${}_3P_3 = 3!$

When we are only required to take a portion of the group of objects, and arrange them we use the **permutation notation** ${}_nP_r$.

$${}_nP_r = \frac{n!}{(n-r)!}$$

${}_n\mathbf{P}_r$ means you have n objects, and you will include r of them in the arrangement or permutation.

n is the **total** number of objects
r is the number of objects chosen to arrange.

the formula for ${}_nP_r$ is $\displaystyle {}_nP_r = \frac{n!}{(n-r)!}$

Using Our Graphing Calculator to Calculate Permutations

Click here to watch a video on how to use the calculator to do Permutations.

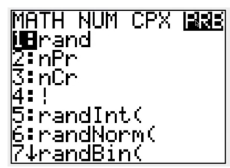
Example:

Calculate ${}_4P_2$.

Press 4 to input the n value of the permutation.

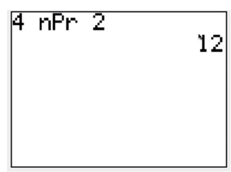
Press MATH and cursor over to PRB

Then press 2: nPr



Now press 2 to input the r value of the permutation

You should get the answer 12.



Why does 0! = 1

In the permutation notation formula $_nP_r = \frac{n!}{(n-r)!}$

we know that n! means the same thing as $_nP_n$. Let's take a look at a particular example 7! $= {_7P_7}$

We know that 7! = 5040

Now lets take a look at $_7P_7$ using the $_nP_r$ formula. In this case, both the n and the r equal 7. Therefore

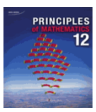
$$_nP_r = \frac{n!}{(n-r)!}$$

$$_7P_7 = \frac{7!}{(7-7)!}$$

$$= \frac{5040}{0!}$$

Since you are not allowed to divide by 0 (that would make the value undefined), we assign the value 0! as value of 1, so that the value of $_7P_7$ is defined

Solving Permutation Problems Where Only Some of the Objects are Used in Each Arrangement



Please turn to page 85 and 86 of the textbook and follow Example 1.

For Example:

How many different three-letter words can be formed from the letters, C, D, E, F, G, and O if not letter is repeated?

Solution:

The first letter can be chosen from any one of the six letters. Having filled this position, the second letter is chosen from the remaining five letters, and consequently, the third letter is chosen from the remaining four letters.

The number of three-letter words is:

$\underline{\ 6\ }\quad \underline{\ 5\ }\quad \underline{\ 4\ }$

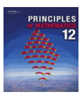
6 x 5 x 4 = 120 three letter words

OR we can use the permutation formula.

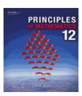
${}_6P_3$ = 120 three letter words.

Solving Permutation Problems With Conditions

There are many different conditions that can happen when dealing with arrangements. In order to be successful in this lesson you should look at all the different types of conditions below.



Please turn to page 88 of the textbook and follow Example 3.



Please turn to page 88 of the textbook and follow Example 4.

For Example:

In how many ways can all of the letters of the ORANGES be arranges if:

a) There are no restrictions?

b) First letter must be an N

c) The vowels must be together in any order.

d) The vowels must be together in the order OAE.

Solution:

a) Since ORANGES has 7 letters, there are 7! or 5040 ways of arranging the word ORANGES

b)

$\underline{1}$ $\underline{6}$ $\underline{5}$ $\underline{4}$ $\underline{3}$ $\underline{2}$ $\underline{1}$ = 720

N

only one N to choose from in the word ORANGE

c)

group OAE is now 1 item therefore

OAE R N G S is 5 items and therefore you should draw 5 blanks

$\underline{5}$ $\underline{4}$ $\underline{3}$ $\underline{2}$ $\underline{1}$ = 120

three in the group so multiply by 3!

120 X 3! = 720

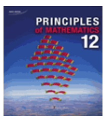
d)

group OAE is now 1 item therefore

OAE R N G S is 5 items and therefore you should draw 5 blanks

$\underline{5}$ $\underline{4}$ $\underline{3}$ $\underline{2}$ $\underline{1}$ = 120

since the vowels must be in the order OAE, you do not have to multiply by 3!. So the final answer is 120



Practice Problem:

Complete “Practising” question 1 on page 93 of your textbook.

Solution:

1. a)
$$
\begin{aligned}
{}_5P_2 &= \frac{5!}{(5-2)!}\\
{}_5P_2 &= \frac{5!}{3!}\\
{}_5P_2 &= \frac{5\cdot4\cdot3!}{3!}\\
{}_5P_2 &= 5\cdot4\\
{}_5P_2 &= 20
\end{aligned}
$$

b)
$$
\begin{aligned}
{}_8P_6 &= \frac{8!}{(8-6)!}\\
{}_8P_6 &= \frac{8!}{2!}\\
{}_8P_6 &= \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2!}{2!}\\
{}_8P_6 &= 8\cdot7\cdot6\cdot5\cdot4\cdot3\\
{}_8P_6 &= 20160
\end{aligned}
$$

c)
$$
\begin{aligned}
{}_{10}P_5 &= \frac{10!}{(10-5)!}\\
{}_{10}P_5 &= \frac{10!}{5!}\\
{}_{10}P_5 &= \frac{10\cdot9\cdot8\cdot7\cdot6\cdot5!}{5!}\\
{}_{10}P_5 &= 10\cdot9\cdot8\cdot7\cdot6\\
{}_{10}P_5 &= 30240
\end{aligned}
$$

d) $_{9}P_{0}=\frac{9!}{(9-0)!}$

$_{9}P_{0}=\frac{9!}{9!}$

$_{9}P_{0}=1$

e) $_{7}P_{7}=\frac{7!}{(7-7)!}$

$_{7}P_{7}=\frac{7!}{0!}$

$_{7}P_{7}=\frac{7!}{1}$

$_{7}P_{7}=7!$

$_{7}P_{7}=7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$

$_{7}P_{7}=5040$

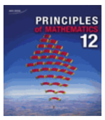
f) $_{15}P_{5}=\frac{15!}{(15-5)!}$

$_{15}P_{5}=\frac{15!}{10!}$

$_{15}P_{5}=\frac{15\cdot 14\cdot 13\cdot 12\cdot 11\cdot 10!}{10!}$

$_{15}P_{5}=15\cdot 14\cdot 13\cdot 12\cdot 11$

$_{15}P_{5}=360360$



Practice Problem:

Complete “Practising” question 2 on page 93 of your textbook.

Solution:

2. a) e.g.,

Permutation	**President**	**Vice-President**
1	Katrina	Jess
2	Katrina	Nazir
3	Katrina	Mohamad
4	Jess	Katrina
5	Jess	Nazir
6	Jess	Mohamad
7	Nazir	Jess
8	Nazir	Katrina
9	Nazir	Mohamad
10	Mohamad	Nazir
11	Mohamad	Jess
12	Mohamad	Katrina

There are 12 different ways that a president and vice-president can be elected.

b) $$_nP_r = \frac{n!}{(n-r)!}$$

$$_4P_2 = \frac{4!}{(4-2)!}$$

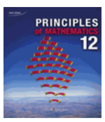
$$_4P_2 = \frac{4!}{2!}$$

$$_4P_2 = \frac{4 \cdot 3 \cdot 2!}{2!}$$

$$_4P_2 = 4 \cdot 3$$

$$_4P_2 = 12$$

The formula for $_nP_r$ gives an answer of 12. This matches my results from part a).



Practice Problem:

Complete “Practising” question 5 on page 93 of your textbook.

Solution:

5. $_9P_3 = \dfrac{9!}{(9-3)!}$

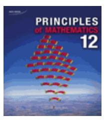
$_9P_3 = \dfrac{9!}{6!}$

$_9P_3 = \dfrac{9 \cdot 8 \cdot 7 \cdot 6!}{6!}$

$_9P_3 = 9 \cdot 8 \cdot 7$

$_9P_3 = 504$

There are 504 different ways the positions can be filled.



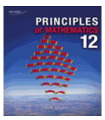
Practice Problem:

Complete “Practising” question 7 on page 93 of your textbook.

Solution:

7. $${}_8P_8 = \frac{8!}{(8-8)!}$$
$${}_8P_8 = \frac{8!}{0!}$$
$${}_8P_8 = \frac{8!}{1}$$
$${}_8P_8 = 8!$$
$${}_8P_8 = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$${}_8P_8 = 40320$$

Therefore, 40 320 different signals could be created.



Practice Problem:

Complete “Practising” question 9 on page 93 of your textbook.

Solution:

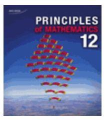
9. e.g., Assuming that any of the 10 digits can be put in any of the 8 remaining spots for the SINs, let S represent the number of social insurance numbers:

$S = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

$S = 10^8$

$S = 100\ 000\ 000$

There are 100 000 000 different SINs that can be registered in each of these groups of provinces and territories.



Practice Problem:

Complete "Practising" question 10 on page 94 of your textbook.

Solution:

10. a)

$$_{12}P_5 = \frac{12!}{(12-5)!}$$

$$_{12}P_5 = \frac{12!}{7!}$$

$$_{12}P_5 = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{7!}$$

$$_{12}P_5 = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$$

$$_{12}P_5 = 95040$$

There are 95 040 ways the coach can select the starting five players.

b)

$$_{11}P_4 = \frac{11!}{(11-4)!}$$

$$_{11}P_4 = \frac{11!}{7!}$$

$$_{11}P_4 = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{7!}$$

$$_{11}P_4 = 11 \cdot 10 \cdot 9 \cdot 8$$

$$_{11}P_4 = 7920$$

There are 7920 ways the coach can select the starting five players, if the tallest student must start at the centre position.

c) $_{10}P_3 = \frac{10!}{(10-3)!}$

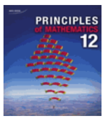
$_{10}P_3 = \frac{10!}{7!}$

$_{10}P_3 = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!}$

$_{10}P_3 = 10 \cdot 9 \cdot 8$

$_{10}P_3 = 720$

Multiply by 2, since Sandy and Natasha can play the guard positions in either order. $720 \times 2 = 1440$
There are 1440 ways in which the coach can select the starting five players, if Sandy and Natasha must play the two guard positions.



Practice Problem:

Complete "Practising" question 11 on page 94 of your textbook.

Solution:

11. a) $n \geq 0$ and

$n - 1 \geq 0$

$n \geq 1$

Therefore, the expression is defined for $n \geq 1$, where $n \in$ I.

b) $n + 2 \geq 0$

$n \geq -2$

Therefore, the expression is defined for $n \geq -2$, where $n \in$ I.

c) $n + 1 \geq 0$ AND $n \geq 0$

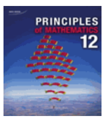
$n \geq -1$

Therefore, the expression is defined for $n \geq 0$, where $n \in$ I.

d) $n + 5 \geq 0$ AND $n + 3 \geq 0$

$n \geq -5$ $\quad$ $n \geq -3$

Therefore, the expression is defined for $n \geq -3$, where $n \in$ I.



Practice Problem:

Complete “Practising” question 13 on page 94 of your textbook.

Solution:

13. a) $${}_{20}P_5 = \frac{20!}{(20-5)!}$$

$${}_{20}P_5 = \frac{20!}{15!}$$

$${}_{20}P_5 = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15!}{15!}$$

$${}_{20}P_5 = 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16$$

$${}_{20}P_5 = 1860480$$

There are 1 860 480 different ways to award the scholarships.

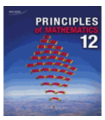
b) Let L represent the number of ways:

$L = 20 \cdot 20 \cdot 20 \cdot 20 \cdot 20$

$L = 20^5$

$L = 3\ 200\ 000$

There are 3 200 000 different ways to award the scholarships.



Practice Problem: (Key Question)

Complete “Practising” question 14 on page 94 of your textbook.

Solution:

14. a) $${}_{10}P_4 = \frac{10!}{(10-4)!}$$

$${}_{10}P_4 = \frac{10!}{6!}$$

$${}_{10}P_4 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!}$$

$${}_{10}P_4 = 10 \cdot 9 \cdot 8 \cdot 7$$

$${}_{10}P_4 = 5040$$

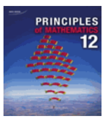
There are 5040 different phone numbers possible.

b) Subtract the total possible numbers by the answer to part a).

104 = 10 000

10 000 – 5040 = 4960

There are 4960 different phone numbers.



Practice Problem:

Complete "Practising" question 15 on page 94 of your textbook.

Solution:

15. a) I need to solve $\frac{n!}{(n-2)!} = 20$.

$n \geq 0$ AND $n - 2 \geq 0$

$n \geq 2$

Therefore, $\frac{n!}{(n-2)!} = 20$ is defined for $n \geq 2$, where $n \in$ I.

$$\frac{n!}{(n-2)!} = 20$$

$$\frac{(n)(n-1)(n-2)(n-3)...(3)(2)(1)}{(n-2)(n-3)...(3)(2)(1)} = 20$$

$$\frac{(n)(n-1)(n-2)!}{(n-2)!} = 20$$

$$(n)(n-1) = 20$$

$$n^2 - n = 20$$

$$n^2 - n - 20 = 0$$

$$(n+4)(n-5) = 0$$

$n + 4 = 0$ or $n - 5 = 0$

$n = -4$ $\qquad$ $n = 5$

The root n = –4 is not a solution to $n \geq 2$.

Check n = 5

LS	RS
${}_5P_2$	20
$\dfrac{5!}{(5-2)!}$	
$\dfrac{5!}{3!}$	
$\dfrac{5 \cdot 4 \cdot 3!}{3!}$	
$5 \cdot 4$	
20	

There is one solution, n = 5.