Math 30-2: U2L2 Teacher Notes

Introducing Permutations and Factorial Notation

Key Math Learnings:

By the end of this lesson, you will learn the following concepts:

4.1 Represent and solve counting problems, using a graphic organizer.

4.4 Solve a contextual counting problem, using the fundamental counting principle, and explain the reasoning.

5.1 Represent the number of arrangements of n elements taken n at a time, using factorial notation.

5.2 Determine, with or without technology, the value of a factorial.

5.3 Simplify a numeric or an algebraic fraction that contains factorials in both the numerator and denominator.

5.4 Solve an equation that involves factorials.

What is the Difference Between a Permutation and Combination?

Let's take a look at two different situations:

A. The combination lock for your bicycle is 26 - 10 - 31 . The order you put these numbers in is very important. If you were to use these same three numbers but mix up the order, you would not be able to open the lock. **In this case, order is important.**

B. When you play a game of cards and in your hand you have 3 kings and 2 queens. This is called a full house. Generally, when you are playing cards, it does not matter what order you hold the cards, all that matters is what cards you are holding. **So in this case, order is not important.**

In this unit, we will be studying two types of combinatorics: **permutations and combinations**.

- The first case in **which order does matter is called a permutation**.
- The second case in **which order does not matter is called a combination**. (We will look at combinations in future lessons)

- Click the icon to watch a YouTube video on the difference between Permutations and Combinations.

Let's Explore Permutations

A **permutation** is an arrangement of objects in a DEFINITE order.

Let’s take a look at all of the ways of arranging the letters of the word CAT.

CAT TCA ATC

CTA TAC ACT

As you can see, there are 6 distinct arrangements of permutations of the letters of the word CAT. By changing the order of the letters you have a different permutation.

If we were to use **Fundamental Counting Principle t**o find the number of permutations of the word CAT, you would make 3 blank lines (one for each letter) and fill them in accordingly.

3 x 2 x 1 = 6 distinct arrangements

In calculating permutations, we often see expressions such as 3 x 2 x 1. **The product of consecutive natural numbers, in decreasing order down to the number 1, can be represented using factorial notation**

$$3 \times 2 \times 1 = 3!$$

(Note: We will discuss permutations at length in Lesson 3)

Factorial Notation (!)

The ! sign is the **factorial sign** and 3! is read as "3 factorial"

Here are some other examples of factorial notation:

6! = 6 x 5 x 4 x 3 x 2 x 1 = 720

5! = 5 x 4 x 3 x 2 x 1 = 120

4! = 4 x 3 x 2 x 1 = 24

3! = 3 x 2 x 1 = 6

2! = 2 x 1 = 2

1! = 1

Here is a peculiar fact: **0! = 1**

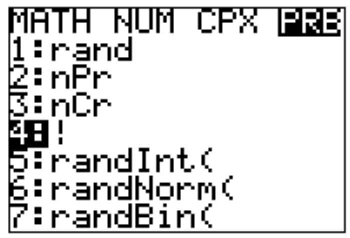
The reason for this will be discussed later in the lesson, when we study permutation notation, we will revisit this fact.

Click the icon to link to a lesson on a Purple Math lesson on Factorial Notation

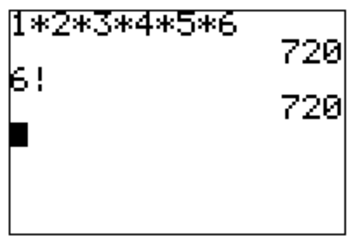
Where Do I Find Factorial Notation in my Calculator?

There is a factorial function on the graphing calculator (and most other mathematical calculators).

Press **MATH** and cursor over to **PRB**.



For example, to **c**alculate 6! we would press 6, MATH, cursor over to PRB and then press 4 or cursor down to ! and press ENTER.



Solving Problems using Factorial Notation

Example:

Simplify $\frac{9!}{4!}$

Solution

$$\frac{9!}{4!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1}$$

Write each factorial notation in its expanded form

$$= \frac{9 \times 8 \times 7 \times 6 \times 5 \times \cancel{4 \times 3 \times 2 \times 1}}{\cancel{4 \times 3 \times 2 \times 1}}$$

Cross out common factors that are in both the numerator and the denominator.

Now evaluate what you are left with $9 \times 8 \times 7 \times 6 \times 5 = 15120$

Therefore $\frac{9!}{4!} = 15120$

Or here's a shortcut for the above question:

In the first step, rather than expanding out both numbers, **choose the bigger number,** and expand that number until you reach the smaller number.

For example, since in

$$\frac{9!}{4!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4!}$$

the 9 is the bigger number, expand 9! until you get to 4! (as follows). Now since you have a 4! on the top and the bottom, you can cancel both out.

$$\frac{9 \times 8 \times 7 \times 6 \times 5 \times \cancel{4!}}{\cancel{4!}}$$

$$9 \times 8 \times 7 \times 6 \times 5$$

$$15120$$

Example:

Simplify $\frac{\mathbf{n!}}{\mathbf{(n-1)!}}$

Solution

Choose the bigger number, (n) and expand that number until you reach the smaller number.

Now since you have a (n - 1)! on the top and the bottom, you can cancel both out.

$$\frac{(n)(n-1)!}{(n-1)!}$$

$$\frac{(n)\,\cancel{(n-1)!}}{\cancel{(n-1)!}}$$

$$n$$

Example:

Simplify $\frac{(n+3)!}{(n+1)!}$

Solution

Choose the bigger number, (n + 3) and expand that number until you reach the smaller number.

Now since you have a (n + 1)! on the top and the bottom, you can cancel both out.

$$\frac{(n+3)(n+2)(n+1)!}{(n+1)!}$$

$$\frac{(n+3)(n+2)\cancel{(n+1)}!}{\cancel{(n+1)}!}$$

$$(n+3)(n+2)$$

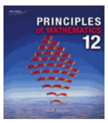
In Summary -- n!

If you have n objects, the number of arrangements or permutations of n different objects is as follows:

n x (n-1) x (n-2) x (n-3) x ...... x 3 x 2 x 1

which basically means that if you have n items, multiply n by every integer from n, down to 1. This is known as factorial notation, and n! is read as “n factorial”.

Also n! is only defined if n is a whole number.
This means numbers like 1.5! and (-2)! are undefined.



Practice Problem:

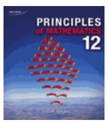
Complete “Check your Understanding” question 2 on page 73 of your textbook.

Solution:

2. a) e.g., There are six permutations of Ken, Sarah, and Raj. I figured this out by making a table showing each permutation and the three possible positions.
Also, $3 \cdot 2 \cdot 1 = 6$

	Position 1	**Position 2**	**Position 3**
Permutation 1	Ken	Sarah	Raj
Permutation 2	Ken	Raj	Sarah
Permutation 3	Sarah	Ken	Raj
Permutation 4	Sarah	Raj	Ken
Permutation 5	Raj	Ken	Sarah
Permutation 6	Raj	Sarah	Ken

b) Let L represent the total number of permutations:
$L = 3 \cdot 2 \cdot 1$
$L = 3!$



Practice Problem:

Complete “Check your Understanding” question 3 on page 73 of your textbook.

Solution:

3. a) $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$

b) $9 \cdot 8 \cdot 7 = \dfrac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

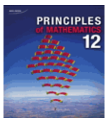
$9 \cdot 8 \cdot 7 = \dfrac{9!}{6!}$

c) $\dfrac{15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1} = \dfrac{15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1} \cdot \dfrac{12!}{12!}$

$\dfrac{15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1} = \dfrac{15!}{12!4!}$

d) $100 \cdot 99 = 100 \cdot 99 \cdot \dfrac{98!}{98!}$

$100 \cdot 99 = \dfrac{100!}{98!}$



Practice Problem:

Complete "Check your Understanding" question 5 on page 73 of your textbook.

Solution:

5. a) $8 \cdot 7 \cdot 6! = 8 \cdot 7 \cdot (6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$

$8 \cdot 7 \cdot 6! = 8 \cdot 7 \cdot 720$

$8 \cdot 7 \cdot 6! = 56 \cdot 720$

$8 \cdot 7 \cdot 6! = 40320$

b) $\dfrac{12!}{10!} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$\dfrac{12!}{10!} = 12 \cdot 11 \cdot \dfrac{10}{10} \cdot \dfrac{9}{9} \cdot \dfrac{8}{8} \cdot \dfrac{7}{7} \cdot \dfrac{6}{6} \cdot \dfrac{5}{5} \cdot \dfrac{4}{4} \cdot \dfrac{3}{3} \cdot \dfrac{2}{2} \cdot \dfrac{1}{1}$

$\dfrac{12!}{10!} = 12 \cdot 11 \cdot \dfrac{10!}{10!}$

$\dfrac{12!}{10!} = 12 \cdot 11 \cdot 1$

$\dfrac{12!}{10!} = 132$

c) $$\frac{8!}{2!\cdot 6!}=\frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(2\cdot 1)\cdot(6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)}$$
$$\frac{8!}{2!\cdot 6!}=\frac{8}{2}\cdot\frac{7}{1}\cdot\frac{6}{6}\cdot\frac{5}{5}\cdot\frac{4}{4}\cdot\frac{3}{3}\cdot\frac{2}{2}\cdot\frac{1}{1}$$
$$\frac{8!}{2!\cdot 6!}=\frac{8}{2}\cdot 7\cdot\frac{6!}{6!}$$
$$\frac{8!}{2!\cdot 6!}=\frac{8}{2}\cdot 7\cdot 1$$
$$\frac{8!}{2!\cdot 6!}=4\cdot 7$$
$$\frac{8!}{2!\cdot 6!}=28$$

d) $$\frac{7\cdot 6!}{5!}=\frac{7\cdot(6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)}{5\cdot 4\cdot 3\cdot 2\cdot 1}$$
$$\frac{7\cdot 6!}{5!}=7\cdot 6\cdot\frac{5}{5}\cdot\frac{4}{4}\cdot\frac{3}{3}\cdot\frac{2}{2}\cdot\frac{1}{1}$$
$$\frac{7\cdot 6!}{5!}=7\cdot 6\cdot\frac{5!}{5!}$$
$$\frac{7\cdot 6!}{5!}=7\cdot 6\cdot 1$$
$$\frac{7\cdot 6!}{5!}=42$$

e) $4\left(\frac{6!}{2!\cdot 2!}\right)=4\left[\frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(2\cdot 1)\cdot(2\cdot 1)}\right]$

$4\left(\frac{6!}{2!\cdot 2!}\right)=4\left(\frac{6}{2}\cdot\frac{5}{1}\cdot 4\cdot 3\cdot\frac{2}{2}\cdot\frac{1}{1}\right)$

$4\left(\frac{6!}{2!\cdot 2!}\right)=4\left(\frac{6}{2}\cdot 5\cdot 4\cdot 3\cdot\frac{2!}{2!}\right)$

$4\left(\frac{6!}{2!\cdot 2!}\right)=4\left(\frac{6}{2}\cdot 5\cdot 4\cdot 3\cdot 1\right)$

$4\left(\frac{6!}{2!\cdot 2!}\right)=4(3\cdot 5\cdot 4\cdot 3)$

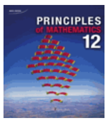
$4\left(\frac{6!}{2!\cdot 2!}\right)=4(180)$

$4\left(\frac{6!}{2!\cdot 2!}\right)=720$

f) $4!+3!+2!+1!=(4\cdot 3\cdot 2\cdot 1)+(3\cdot 2\cdot 1)+(2\cdot 1)+1$

$4!+3!+2!+1!=24+6+2+1$

$4!+3!+2!+1!=33$



Practice Problem:

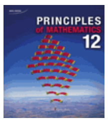
Complete “Check your Understanding” question 7 on page 74 of your textbook.

Solution:

7. There are nine students in the lineup, so there are nine possible positions. Let L represent the total number of permutations:

$L = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 9!$
$L = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 72 \cdot 7 \cdot 30 \cdot 4 \cdot 6$
$L = 72 \cdot 210 \cdot 24$
$L = 362\ 880$

There are 362 880 permutations for the nine students at the Calgary Stampede.



Practice Problem:

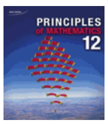
Complete “Check your Understanding” question 8 on page 74 of your textbook.

Solution:

8. There are five students in the club and there are five possible positions. Let L represent the total number of permutations:

$L = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 5!$
$L = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 20 \cdot 6$
$L = 120$

There are 120 different ways to select members for the five positions.



Practice Problem:

Complete “Check your Understanding” question 9 on page 74 of your textbook.

Solution:

9. There are six activities to do and there are six days.
Let L represent the total number of permutations:

$L = 6!$

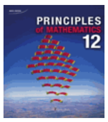
$L = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$L = 30 \cdot 4 \cdot 6$

$L = 120 \cdot 6$

$L = 720$

There are 720 different ways they can sequence these activities over the six days.

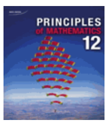


Practice Problem:

Complete “Check your Understanding” question 15 on page 75 of your textbook.

Solution:

15. There would be 7 chuckwagons behind Brant’s so there are 7 spots where the other drivers could finish.
Let L represent the number of permutations:
$L = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 7!$
$L = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 42 \cdot 20 \cdot 6$
$L = 42 \cdot 120$
$L = 5040$
If Brant’s wagon wins, there are 5040 different orders in which the eight chuckwagons can finish.



Practice Problem:

Complete “Check your Understanding” question 16 on page 75 of your textbook.

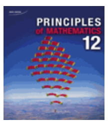
Solution:

16. a) e.g., YKONU, YUKNO, YKNOU
b) There are five letters so there are five spots to put the letters. Let L represent the number of permutations:

$L = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$L = 5!$

There are 5! possible permutations. This makes sense because e.g., the integer in the factorial (5 in this case) for the number of permutations is normally equal to the number of spots in which there are things to place. There are five spots to place the letters so this means that the number of permutations should be 5! which matches the answer that was found.



Practice Problem:

Complete “Check your Understanding” question 17 on page 75 of your textbook.

Solution:

17. a) e.g., Using trial and error, I have the following calculations:
$1! = 1$, $2^1 = 2$; $2! = 2$, $2^2 = 4$;
$3! = 6$, $2^3 = 8$; $4! = 24$, $2^4 = 16$
I notice that for $n = 4$, $n!$ is greater than 2^n. This continues for $n \geq 4$ because 2^4 will keep getting multiplied by 2, while 4! will keep getting multiplied by numbers greater than 2 to obtain the higher factorials.
b) e.g., Using what I have in a), I know that for $n < 4$, $n!$ is not greater than 2^n. The calculations for these values of n are shown in a). Thus for $n = \{1, 2, 3\}$, $n!$ is less than 2^n.