Math 30-2: U8L4 Teacher Notes

The Equations of Sinusoidal Functions

Key Math Learnings:

By the end of this lesson, you will learn the following concepts:

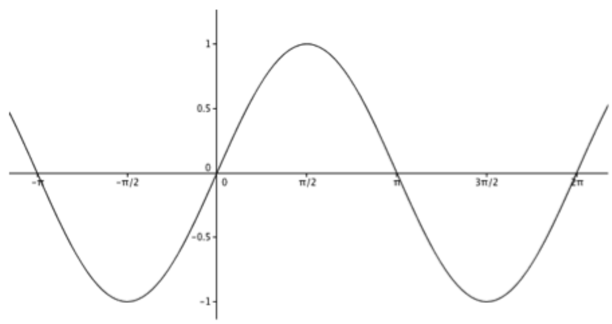
- ★ Describe, orally and in written form, the characteristics of sinusoidal functions by analyzing their graphs.
- ★ Describe, orally and in written form, the characteristics of sinusoidal functions by analyzing their equations.
- ★ Match equations in a given set to their corresponding graphs

How are the Parameters of Equation related to the Characteristics of the Graph

A sine function can be written in the **standard form** $y = a \sin [b(x - c)] + d$, where a, b, c and d, represent real numbers. The values a, b, c and d are called **parameters** of this function while x is a **variable**.

A **parameter** can be defined as a value that is already "built in" to a function. Parameters can be changed so that the function can be used to model different applications.

In lesson 1 we learned that the basic sine function, $y = \sin x$, has the parameters $a = 1$, $b = 1$, $c = 0$ and $d = 0$ and graphically has 5 main points throughout one period as shown to the right.



What happens to the graph when the parameters are different then those of the basic sine function?

Open up the Lesson 4 Investigation and complete it before continuing on to the rest of the notes.

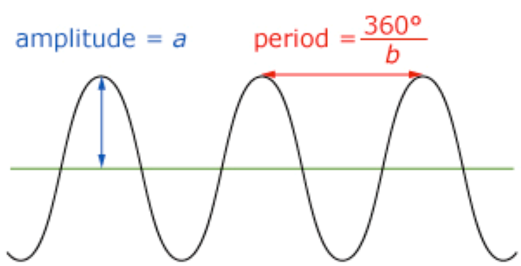
Parameters *a* and *b*

In the Lesson 4 Investigation, you explored how *a, b, c* and *d* are related to characteristics of the graph $y = a \sin [b(x - c)] + d$ with the angle measured in radians.

The amplitude of the graph is determined by the parameter *a* of the function.

The parameter *b* is related to the period by the equation $P = \frac{360°}{b}$ where *P* represents the period and the angle is measured in degrees.

Notice that a large *b*-value results in a shorter period, and a small *b*-value results in a longer period.



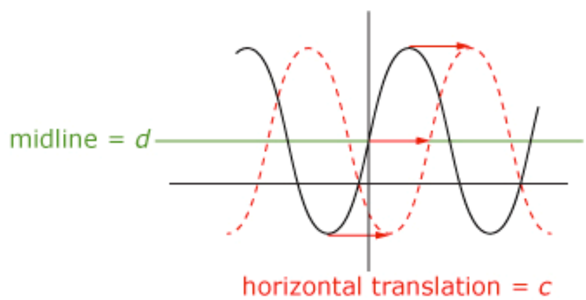
Parameters *c* and *d*

The parameters *c* and *d* do not change the shape of the graph, but they do move the graph vertically and horizontally.

When the graphs are moved up or down, three characteristics are affected: the midline, the maximum, and the minimum.

- The midline will occur at $y = d$.
- The maximum value = $d + a$.
- The minimum value = $d - a$.

In the Investigation you noticed that changing the parameter *c* results in the graph being shifted *c* units horizontally.



Inserting a positive *c*-value into $y = a \sin b(x - c) + d$ makes the *c* term appear negative.

$c = 5$ in $y = a \sin b(x - 5) + d$ is a translation of 5 units to the right of $y = \sin x$.

$c = -5$ in $y = a \sin b(x - (-5)) + d$, which is equal to $y = a \sin b(x + 5) + d$, is a translation of 5 units to the left of

$y = \sin x$.

How do Parameters Affect The Characteristics of the Cosine Graph

The parameters *a*, *b*, *c*, and *d* in $y = a \cos[b(x - c)] + d$ describe the same characteristics as they do for $y = a \sin[b(x - c)] + d$. So far, you have worked with sine and cosine functions in degrees. It is also possible to use an *x*-value in radians. All the parameters behave the same, but now *c* is in radians.

The relationship between *b* and the period also changes slightly: $360° = 2\pi$, so use $P = \frac{2\pi}{b}$ instead of $P = \frac{360°}{b}$ when working in radians.

The following example shows how an equation in radians can be interpreted.

Example:

Harry examined the function $y = \frac{1}{2}\cos 3(x+4)+2$ and asked the following questions.

a. What is the equation of the midline?

b. What is the period of the graph?

c. Describe any horizontal translation of the graph from *y* = cos *x*.

- d. What are the maximum and minimum values?

Solution:

a. The midline has the equation $y = d$. This means the equation of the midline is $y = 2$.

b. The equation is written in radians because the *c*-value has no unit. 360° = 2*π* rad. So, $P = \frac{2\pi}{b}$ can be used to determine the period of the function.

$$\begin{aligned} P &= \frac{2\pi}{b} \\ &= \frac{2\pi}{3} \\ &\approx 2.09 \end{aligned}$$

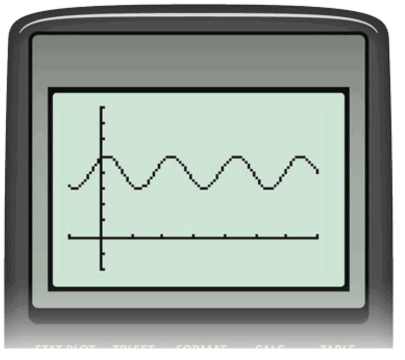
c. The *c*-value determines any horizontal translation from $y = \cos x$.

A negative four must have been substituted for *c* into $y = a \sin [b(x - c)] + d$. This means the graph moved left 4 units.

d. The amplitude is equal to *a* and so is 1/2. The maximum occurs one amplitude above the midline, so the maximum is $2 + \frac{1}{2} = 2.5$.

The minimum occurs one amplitude below the midline, so the minimum is $2 - \frac{1}{2} = 1.5$.

The graphing calculator can graph $y = \frac{1}{2}\cos 3(x + 4) + 2$ in order to check the solutions.



Determining the Equation of a Function Given its Graph

Sometimes it is useful to determine the equation of a function given its graph. The following video describes this process.

Click the icon to watch a video on determining the equation of a function given its graph

From the video we can summarize the steps for determining the equation of the function from its graph

Step 1: Determine the maximum and minimum values of the graph. Use these values to determine the midline and the amplitude. This is the *a* and the *d* of the function.

$$\text{Midline} = \frac{\max + \min}{2} \quad \text{and Amplitude} = \frac{\max - \min}{2}$$

Step 2: Determine the period of the graph. The parameter *b* can be calculated by the formula

$$b = \frac{2\pi}{P} \quad \text{or} \quad b = \frac{360^\circ}{P}$$

Step 3: Determine the *c* by choosing the function *y* = sin *x* or *y* = cos *x* first.

- If *y* = sin *x*, then *c* is the distance away from *y* -axis and the intersection of the graph with the midline where the graph is increasing.

- If *y* = cos *x*, then *c* is the distance away from the *y* -axis and the maximum point.

Step 4: Substitute the parameters into the standard form of the function.

Applications of Sinusoidal Functions

When a sinusoidal function is used to model a situation, the parameters *a*, *b*, *c*, and *d* can be interpreted to provide information about the model.

Example:

The height of a swing over time can be modelled by the function

$$h(t) = 15\cos\left(\frac{2\pi}{3}t\right) + 65$$

where *h* is the height in centimetres above the ground and *t* is the time in seconds.

a. What is the highest point the girl in the photo will reach?

b. Determine the height of the girl at 3.5 s and at 8.0 s.

c. In terms of the movement of the swing, explain what the 65 represents.

d. Explain what the 15 represents in terms of the movement of the swing.

e. Suppose the girl in the photo is at the highest point. How long will it take her to reach that point again?

Solution:

a. The midline occurs at 65 and the amplitude is 15, so the maximum is 65 + 15 = 80. Therefore, 80 cm is the highest the girl will reach.

b.

$$h(t) = 15\cos\left(\frac{2\pi}{3}t\right) + 65$$

$$h(3.5) = 15\cos\left(\frac{2\pi}{3}(3.5)\right) + 65$$

$$= 72.5$$

She will be 72.5 cm high at 3.5 s.

$$h(t) = 15\cos\left(\frac{2\pi}{3}t\right) + 65$$

$$h(8.0) = 15\cos\left(\frac{2\pi}{3}(8.0)\right) + 65$$

$$= 57.5$$

She will be 57.5 cm high at 8 s.

c. The 65 represents the midline of the graph of the function. This is halfway between the highest and lowest point the girl will reach.

d. The 15 represents the amplitude of the graph of the function. This means the girl will travel 15 cm above the midline and 15 cm below the midline.

e. The girl will return to the highest point again after two periods. (She will reach a maximum on the other side of the swing before returning.)

$$\begin{aligned} P &= \frac{2\pi}{b} \\ &= \frac{2\pi}{\left(\frac{2\pi}{3}\right)} \\ &= 3 \end{aligned}$$

So, the girl will take 6 s to return to this position.

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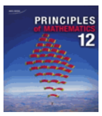


Practice Problem:

Complete “Check your Understanding” question 5 on page of your textbook.

Solution:

5. a) Amplitude = 7

Minimum value $= d - a$

Minimum value $= 0 - 7$

Minimum value $= -7$

Maximum value $= d + a$

Maximum value $= 0 + 7$

Maximum value $= 7$

The amplitude of this function is 7, and the range is $\{y \mid -7 \le y \le 7, y \in \mathrm{R}\}$.

b) Amplitude = 13

Minimum value $= d - a$

Minimum value $= 0 - 13$

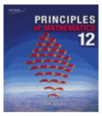
Minimum value $= -13$

Maximum value $= d + a$

Maximum value $= 0 + 13$

Maximum value $= 13$

The amplitude of this function is 13, and the range is $\{y \mid -13 \le y \le 13, y \in \mathrm{R}\}$.



Practice Problem:

Complete "Check your Understanding" question 6 on page of your textbook.

Solution:

6. a) Equation of the midline: $y = 5$

Amplitude = 8

Minimum value = $d - a$

Minimum value = $5 - 8$

Minimum value = -3

Maximum value = $d + a$

Maximum value = $5 + 8$

Maximum value = 13

The equation of the midline of this function is $y = 5$. The amplitude is 8, and the range is $\{y \mid -3 \le y \le 13, y \in R\}$.

b) Equation of the midline: $y = -7$

Amplitude = 6

Minimum value = $d - a$

Minimum value = $-7 - 6$

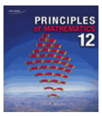
Minimum value = -13

Maximum value = $d + a$

Maximum value = $-7 + 6$

Maximum value = -1

The equation of the midline of this function is $y = -7$. The amplitude is 6, and the range is $\{y \mid -13 \le y \le -1, y \in R\}$.



Practice Problem:

Complete “Check your Understanding” question 7 on page of your textbook.

Solution:

7. a) Equation of the midline: $y = 2$

Maximum value $= d + a$

Maximum value $= 2 + 5$

Maximum value $= 7$

Minimum value $= d - a$

Minimum value $= 2 - 5$

Minimum value $= -3$

The equation of the midline of this function is $y = 2$. The maximum value is 7, and the minimum value is –3.

b) Equation of the midline: $y = -3$

Maximum value $= d + a$

Maximum value $= -3 + 3$

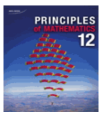
Maximum value $= 0$

Minimum value $= d - a$

Minimum value $= -3 - 3$

Minimum value $= -6$

The equation of the midline of this function is $y = -3$. The maximum value is 0, and the minimum value is –6.

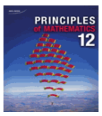


Practice Problem:

Complete “Check your Understanding” question 8 on page of your textbook.

Solution:

8. a) $y = \sin x$ would have to be horizontally translated 30° right.
b) $y = \cos x$ would have to be horizontally translated 100° left.
c) $y = \sin x$ would have to be horizontally translated 4.5 right.
d) $y = \cos x$ would have to be horizontally translated 3 left.

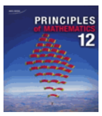


Practice Problem:

Complete “Check your Understanding” question 11 on page of your textbook.

Solution:

11. a) If the equation is in the form $y = a \sin b(x + c) + d$, the amplitude is a, the midline is $y = d$, it is translated to the right by c degrees or radians and the period is $\frac{360°}{b}$ or $\frac{2\pi}{b}$. Thus, for the amplitude and period to be a fourth of the original value, a must be one-fourth the value of the original equation and b must be four times the value of the original equation. Therefore, the function must be $y = 0.5 \cos 4x$.



Practice Problem:

Complete “Check your Understanding” question 13 on page of your textbook.

Solution:

a. The maximum value of this graph is 5, and the minimum value is -1. This eliminates equations iv) and v) because equation iv) has a minimum of -4 and equation v) has a minimum of 0.

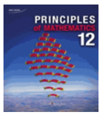
The midline of this graph is y = 2. The graph first intersects the midline and is increasing at x = 120 degrees, and the first maximum is 210 degrees. Therefore, the horizontal translation is 120 degrees for a sine graph and 210 degrees for a cosine graph.

Equation i) is eliminated, because tis horizontal translation is -120 degrees or 240 degrees, and equation ii) is eliminated, because its horizontal translation is 90 degrees or 450 degrees.

b) The maximum value of this graph is 4, and the minimum value is -2. This eliminates equations iv) and v) because equation iv) has a minimum of -4, and equation v) has a minimum of 0.

The midline of this graph is y = 1.

The graph first intersects the midline and is decreasing at x = 60 degrees, and the first minimum is at 150 degrees. Therefore the transilaton is -120 degreees for a sine function and -210 degrees for a cosine function. Equation i) corresponds to thsi graph.



Practice Problem: (KEY QUESTION)

Complete "Check your Understanding" question 15 on page of your textbook.

Solution:

15. a) Range: $\{y \mid -4 \le y \le 2, y \in R\}$
Amplitude = 3
Maximum value = $d + a$
Maximum value = $-1 + 3$
Maximum value = 2
Minimum value = $d - a$
Minimum value = $-1 - 3$
Minimum value = -3
Equation of the midline: $y = -1$

$$\text{Period} = \frac{360°}{b}$$

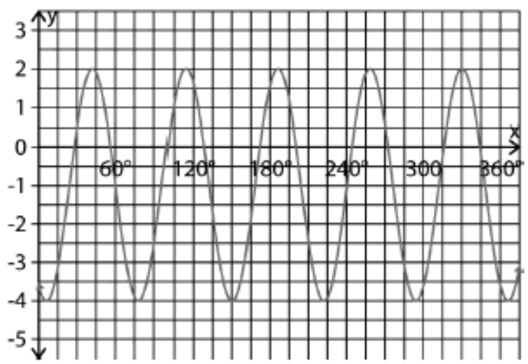
$$\text{Period} = \frac{360°}{5}$$

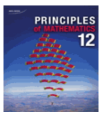
$$\text{Period} = 72°$$

Horizontal translation = $-30°$
The range of this function is $\{y \mid -4 \le y \le 2, y \in R\}$, and the amplitude is 3. The equation of the midline is $y = -1$. The period is 72°, and the graph has been translated 30° to the left (or $-30°$).

Solution:

b) e.g., I plotted the equation using graphing technology and the graph matched the characteristics.





Practice Problem:

Complete “Check your Understanding” question 21 on page of your textbook.

Solution:

Example: If the equation is in the form $y = a \sin b\,(x - c) + d$, the amplitude is *a*, the midline is $y = d$, it is translated to the right by *c* degrees or radians and the period is $\frac{360°}{b}$ or $\frac{2\pi}{b}$. For example, the equation $y = 2 \sin 2\,(x - 45°) + 3$ has an amplitude of 1, a midline of y = 3, it is translated to the right 45 degrees, and a period of 180 degrees.