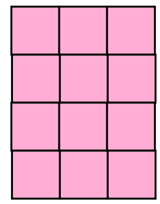
We must help children make sense of the link between multiplication an s groups of and multiplication as arrays or areas.

We assume that they see the relationship but my investigations in classrooms suggest otherwise.

Students constantly see groups of as circles with things in them. IN arrays they must move that picture to linear arangments of rows and columns.

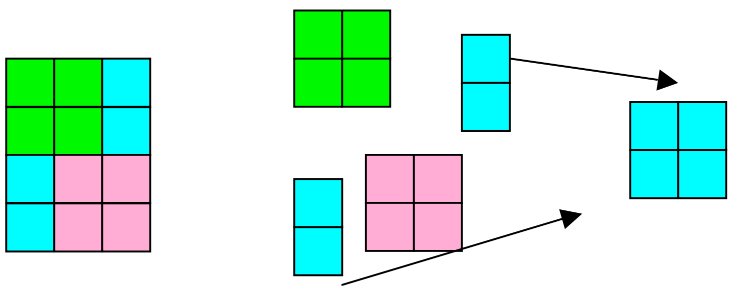
I made the following array and invited students to study it.



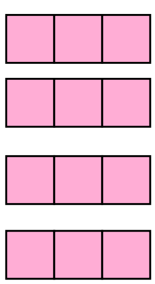
I told them I would label this array as a 3 by 4 and wrote
3 x 4 = 12 on the board.
A voice exclaimed, hey that is just like multiplication 3 times 4 equals 12.
Yes, building arrays is one way to”see” multiplication facts.

To the class :
Your teacher told me he has introduced multiplication as groups of: Can you see 3 groups of 4 in my array?

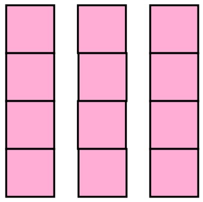
Silence for several minutes then one hand raised, I see it and a little girl came up and separated out 3 groups of 4:



I asked if there was another way to see 3 groups of 4 and a student came up and did this:



When I asked him to point out 1 group of 4, he was lost so I circled one of his groups and asked how many were in it? Three. How many groups have you made? 4..... So you have ___ groups of _______ .
So you showed us 4 groups of 3 are in this.... does someone see how 3 groups of 4 is also here?



So when you build an array you can take it apart and put it together in two different directions.
You can pull it apart horizontally, in rows (pull it apart into 3's as above) to show 4 groups of 3.
Push it back together: And 4 groups of 3 creates an area of 12.
Or you can pull it apart vertically, in columns and this way you see 3 groups of 4. And 3 groups or 4 still gives you an area of 12.

Build

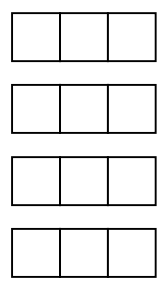
It is very important that students construct meaning for multiplication as dimensional. Multipliciation creates dimensional growth. Factors can be connected to the side lengths of rectangles and later to the dimensions of rectangular prisms.

Invite students to put out 4 groups of 3 tiles.

Typically several if not all students will put out disconnected groups.

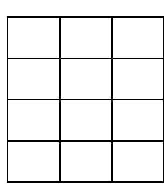
Demonstrate how to move the tiles into an area model.

Here I have created 4 rows with three tiles in each.



What do you think I mean by rows?
(rows run horizontally, across right to left, left to right)

When I push them together they create a rectangle. You push yours together as well.

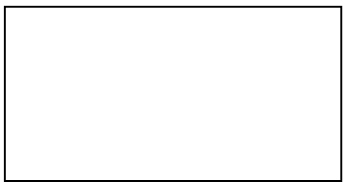


Represent

Help me draw a representation of what I did with the tiles

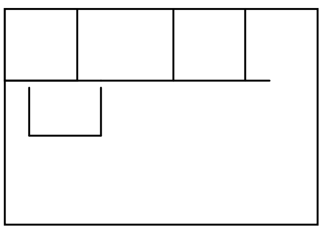
Students may have several suggestions but what is important is to focus their attention on the units of 4 tiles being pushed together.

Which of these suggestions most clearly follows what I did with the tiles?

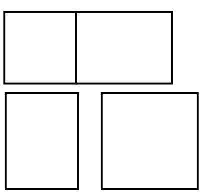


Draw a rectangle

Students who show variations of this method are focused on counting, not units of 3 being collected. Often the squares are mismatched and do not fit into the rectangle.

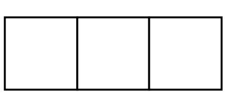


Now fill it with little squares

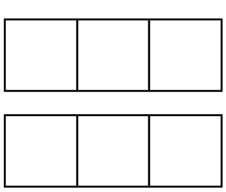


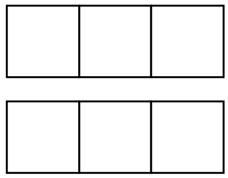
Draw individual tiles, one at a time in any order

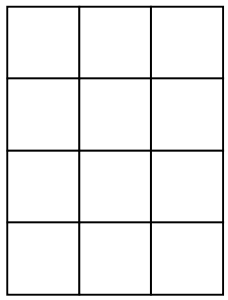
Again the student is focused on counting, not units of 3 being collected.
In this case the count is not related to the threes we collected but an after thought having seen the final rectangle .



Draw a row of 3 tiles, then another under it, then another under it and again







Then draw them pushed together to make the rectangle

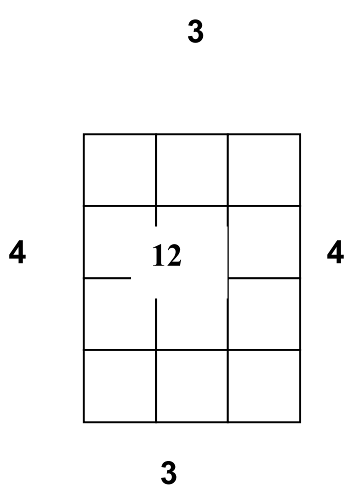
The area of my rectangle is 12 squares.
Do you know what I am counting to name the area?

The tiles, you are counting each tile

Yes and the tiles are square so I am counting the square units or square tiles.

A rectangle that is 3 by 4 is made of 3 rows of 4 tiles each.

I can label my diagram now with 3 and 4 and put on the inside 12.



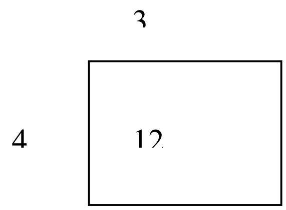
Explain

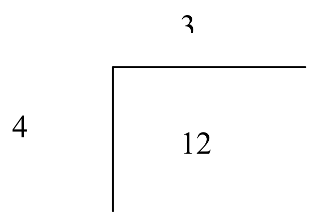
Explain my labels.

The three at the top and bottom means it is three across, you counted 3 acorss, it measures three tiles across

The four at the sides means there are 4 up and down or in each column

If I replace my drawing with this drawing do you understand what my labels are trying to explain?

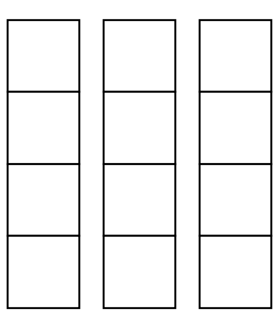




And how about this drawing?

Let’s look back at the model we built.

It was four rows of three but now I can pull it apart to see three columns of four. Watch



If I made my rectangle from 3 columns of 4 each how would I draw what I did?
Do I still have an area of 12 square units or tiles?

Compare

So I can take 12 apart as 3 groups of 4 or 4 groups of 3. Either way I have 12 as my area.

We record this as 3 x 4 = 12 or 4 x 3 = 12.

If I turn the rectangle will I still have 3 x 4 or 4 x 3?
Can I still pull it apart both ways?

Will I still have 12 square tiles or units in my area?

Then I am able to write So 3 x 4 = 4 x 3

What would you put in your book to represent what we just did?

Synthesize

Arrays and area models are linked to groups of. Multiplication is about iterating a unit, not single counting, not adding.
Let's try a couple more:

This time take out five groups of 6 tiles.
Arrange them into a rectangle that shows five groups of six.
Did you make your rectangle with single tiles or with groups of six?

How would we represent this rectangle?
Again focus on drawing from units of six.
Can you see units of five in your rectangle?

This rectangular array is easy to count by 3's or 4's.Can you show me?
If you count the rows you can go 3 + 3 + 3 + 3 and that equals 12 because 6 + 6 = 12 or 3,6,9,12.
Four 3's is 12 or 4 x 3 = 12

How could you count by fours?
If you count the columns you can count 4 + 4 + 4 which is 12 or 4, 8, 12.
Three 4's is 12 or 3 x 4 = 12.

Give students graph paper and direct them to outline this rectangle and cut it out.
Write its two equations on the back. 3 x 4 = 12 4 x 3 = 12

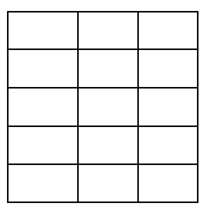
What happens to the area if we build another row on?
(It increases by 3)

What is the new area? (15)

Did you count by 3's? By 5's? Did you add on 3 to the last one?

Label the sides

3

5  5

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How is it different from the first rectangle we made?
This is a 3 x 5 the last one was 3 x 4, It has 3 more, it is the same width but longer, it has one more row

How is it the same?
It has 3 columns, it is three wide

Is it still an area of 15 squares?

5 x 3 = 15

How can we count this area?

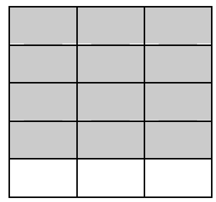
3 + 3 + 3 + 3 + 3 = 15 squares 3,6,9,12,15
5 + 5 + 5 = 15 squares 5,10,15

Three fives is the same as 3 x 5
Five threes is the same as 5 x 3

Draw this one on your graph paper, cut it out and label the back. 3 x 5 = 15 5 x 3 = 15

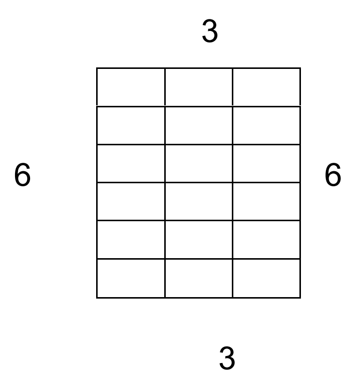
Lay the 3 x 5 on top of the 3 x 4.
How are they the same? How are they different?

3 x 4 = 12
3 x 5 = 15   What do you see?



Go back to the tile rectangle you made on your desk.
Add on another row.
What is the new area? How do we label this one?
3 x 6 = 18 squares.
How did you count it?

3 x 5 + 3= 18
Or you could count by 3's   3+3+3+3+3+3= 18 or
3,6,9,12,15,18
I saw 6 and 6 is twelve and three more is fifteen
I added from 15 to get 16,17,18



If you turn it you have a 6 x 3. What is its area? *18 squares.*

Build this one on graph paper, cut it out and label it on the back 3 x 6= 18 6 x 3 = 18 (Grade 2's match it to the flashcard)

If you put the three we have now built together what do they have in common?
How are they the same, how are they different?
All rectangles, all have 3 columns, all have groups of 3's, all made of squares, areas are different by 3 each time, getting bigger, 12, 15, 18 there is three different in the area each time, the 18 has 6 more than the 12

Noting similarities and differences is how the brain makes sense. This inductive strategy is one of the most powerful instructional strategies and has the potential to reach the greatest number of students. (Marzano, 2001; Bransford, 1999). Students need to investigate all the links between area, arrays, repeated addition and multiplication. Some will hook to the number pattern, some to the spatial arrays. The more students discuss and recognize about these shapes, the more likely they will be to internalize the information in connected networks of meaningful and easily retrieved information.

With the help of students build a chart of the rectangles on a large piece of paper that can stay displayed in the room.

These will be the headings for the chart:

# of columns   # of rows   Area

Have them help you fill in the information so far, as they do encourage them to continue to discuss What stays the same each time?
3 columns, three in each row, 3 across the to, the width

What changes each time?
The number of rows, the total number of tiles, the number of squares, the area and the length, one side gets longer

# of columns	# of rows	Area
3	4	12 squares
3	5	15 squares
3	6	18 squares

Keep building and discussing with students.

# of columns	# of rows	Area
3	3	9 squares
3	4	12 squares
3	5	15 squares
3	6	18 squares

What came before the 3 x 3? Can you see it in your head?
(Our aim is to have students begin to visualize these arrays)

What will we name it? *3 x 2, it has 2 rows with 3 in each, 3 columns with 2 in each*

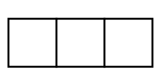
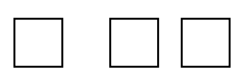
What will the area be?   3 + 3 = 6 two three's

Build it with tiles, on graph paper and add it to the chart

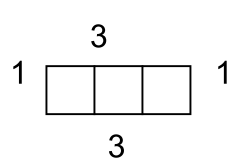
What comes before 3 x 2 = 6???
Students can see the missing one but don't always make sense of why it is 3 x 1

Looking at the chart you can see that 3 x 1 is missing. Building it you can see one row of three. It is sometimes very difficult for students to conceptualize three columns with one in each.
By connecting it to a pattern in the chart we help them see how it naturally arises from the pattern we have recognized.

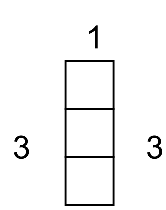
With each succession students will cut out the matching array from graph paper

One row of three  3 columns of 1 

One group of 3



3 groups of 1



# of columns	# of rows	Area
3	1	3 squares
3	2	6 squares
3	3	9 squares
3	4	12 squares
3	5	15 squares
3	6	18 squares

Continue the pattern forward:

So what comes after the 3 x 6? Use the same focused discussion to build through to the 3 x 12.

Students will finish with a set of arrays that are labelled from 3 x 1 = 3 to 3 x 12 = 36.

Lesson Two

Start with a review of yesterday's data. Have students hold up the array from their bag that matches the 3 x 4 which was where we started. Have them describe it.

3 columns, 4 rows, the area is 12 square.

Have someone build it on the overhead, label it 3 x 4 and show how you can take it apart to see
3 columns of 4 tiles 3 x 4 = 12
or 4 rows of 3 each 4 x 3 = 12

Add this information to the chart.

# of columns	**# of rows**	**Area**		
3	1	3 squares		
3	2	6 squares		
3	3	9 squares		
3	4	12 squares	3 x 4 = 12	4 x 3 = 12
3	5	15 squares		
3	6	18 squares		
3	7	21 squares		
3	8	24 squares		
3	9	27 squares		
3	10	30 squares		
3	11	33 squares		
3	12	36 squares		

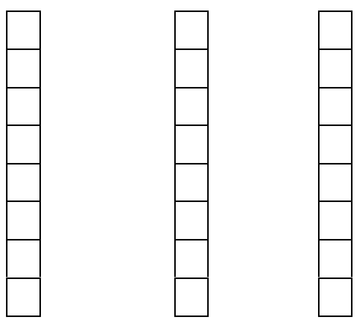
We can add the multiplication facts to our chart

Columns	Rows	Area		
3	1	3 squares	3 x 1 = 3	1 x 3 = 3
3	2	6 squares	3 x 2 = 6	2 x 3 = 6
3	3	9 squares	3 x 3 = 9	3 x 3 = 9
3	4	12 squares	3 x 4 = 12	4 x 3 = 12
3	5	15 squares	3 x 5 = 15	5 x 3 = 15
3	6	18 squares	3 x 6 = 18	6 x 3 = 18
3	7	21 squares	3 x 7 = 21	7 x 3 = 21
3	8	24 squares	3 x 8 = 24	8 x 3 = 24
3	9	27 squares	3 x 9 = 27	9 x 3 = 27
3	10	30 squares	3 x 10 = 30	10 x 3 = 30
3	11	33 squares	3 x 11 = 33	11 x 3 = 33
3	12	36 squares	3 x 12 = 36	12 x 3 = 36

Division:

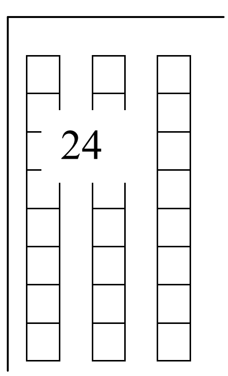
Everyone take out 24 tiles and build the 3 x 8.
What is the area of this array? 24

I can divide this 24 into 3 columns 24 ÷ 3



There will be 8 in each 3

8



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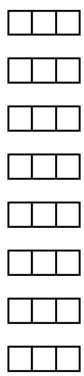
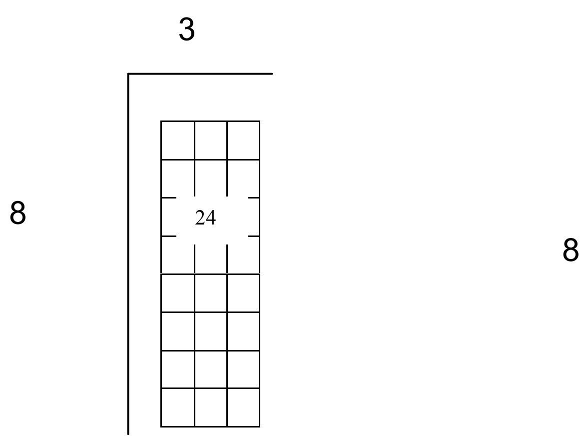
We just made 24 ÷ 3 = 8 which you can also write like this

$$3\overline{)\,24}$$

What does it mean to divide?
We had 24 tiles and we pulled them apart into 3 groups or columns, we split it in 3 groups, the opposite of multiplication, you took apart the rectangle

Or you can divide the 24 into 8 rows, there will be 3 in each

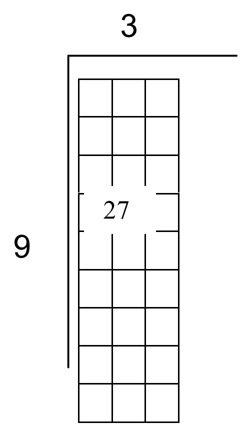




So when you build 3 x 8 = 24
you also have   8 x 3 = 24 (turn the array to show it)
and you have   24 ÷ 3 = 8   (break apart the 3 columns)
and   24 ÷ 8 = 3   (break apart the 8 rows)

Find your 3 x 8 array and put the two division facts on the back.

Try another one. Build the 3 x 9 = 27 with tiles.
Pull apart the 27 into 3 columns with 9 in each. 27 ÷ 3 = 9
Take it apart as 9 groups or rows with 3 in each. 27 ÷ 9 = 3



27 ÷ 9 = 3   27 ÷ 3 = 9

Will each multiplication equation have two division equations to go with it?
Work through the list and the arrays and record them on the chart.
Students can continue to match the flashcards to theirs or record on the back of their arrays.

Columns	**Rows**	**Area**				
3	1	3 squares	3 x 1 = 3	1 x 3 = 3	3 ÷ 3 = 1	3 ÷ 1 = 3
3	2	6 squares	3 x 2 = 6	2 x 3 = 6	6 ÷ 3 = 2	6 ÷ 2 = 3
3	3	9 squares	3 x 3 = 9	3 x 3 = 9	9 ÷ 3 = 3	9 ÷ 3 = 3
3	4	12 squares	3 x 4 = 12	4 x 3 = 12	12 ÷ 3 = 4	12 ÷ 4 = 3
3	5	15 squares	3 x 5 = 15	5 x 3 = 15	15 ÷ 3 = 5	15 ÷ 5 = 3
3	6	18 squares	3 x 6 = 18	6 x 3 = 18	18 ÷ 3 = 6	18 ÷ 6 = 3
3	7	21 squares	3 x 7 = 21	7 x 3 = 21	21 ÷ 3 = 7	21 ÷ 7 = 3
3	8	24 squares	3 x 8 = 24	8 x 3 = 24	24 ÷ 3 = 8	24 ÷ 8 = 3
3	9	27 squares	3 x 9 = 27	9 x 3 = 27	27 ÷ 3 = 9	27 ÷ 9 = 3
3	10	30 squares	3 x 10 = 30	10 x 3 = 30	30 ÷ 3 =10	30 ÷10 = 3
3	11	33 squares	3 x 11 = 33	11 x 3 = 33	33 ÷ 3 =11	33 ÷ 11= 3
3	12	36 squares	3 x 12 = 36	12 x 3 = 36	26 ÷ 3 = 12	36 ÷ 12 = 3

Ask students if they see any patterns?
Be sure whatever students notice repeats several times or it is not a pattern maybe just a similarity between several numbers?

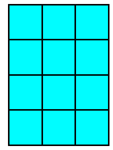
Take out your 3 x 6 = 18 and your 3 x 12 = 36

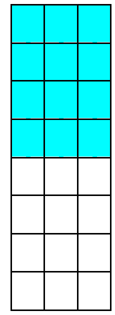
Take out the 3 x 4 = 12 and the 3 x 8 = 24

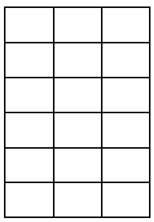
List the equations as pairs on the board.

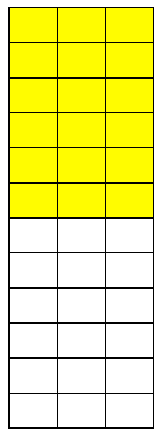
What do you notice about these pairs?
Talk to a partner.
.

3 x 4 = 12
3 x 8 = 24







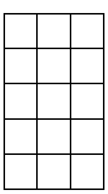


3 x 12 = 36
3 x 6 =

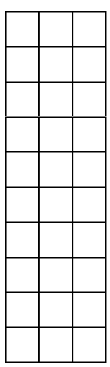
The 3 x 6 is half of the 3 x 12
The 3 x 4 is half of the 3 x 8
They all have 3 columns or a width of 3
6 is half of 12, 12 is half of 24
double 3 is 6, double 12 is 24
double 18 is 36.

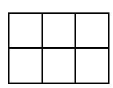
Can you find others that do this? Always compare them visually and write the equations up. We want to deliberately link the pictures to the symbols.

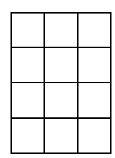
3 x 5 = 30
3 x 10 = 60



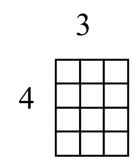
3 x 2 = 6
3 x 4= 12



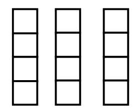




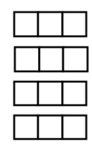
Multiplication as Arrays: Making sense of the multiplication matrix



Build a rectangle that is 3 by 4 or 3 columns of 4 each.
Invite students to show you how they can quickly count how many square tiles make up the rectangle.



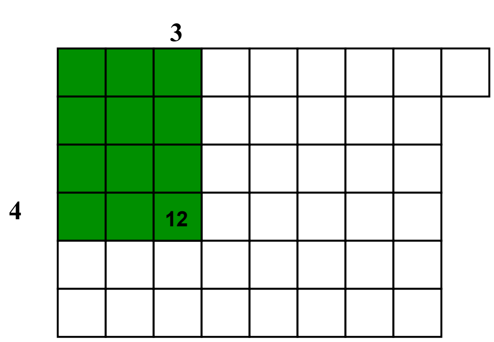
If you separate the columns you can count 4 + 4 + 4 which is 12
Or you can skip count 4, 8, 12.
Three 4's is 12 or 3 x 4 = 12.



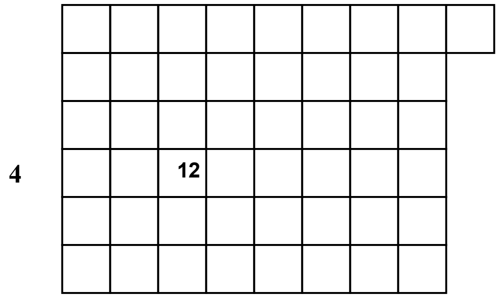
If you separate the rows you can go 3 + 3 + 3 + 3 and that equals 12
Or you can count 3 plus 3 and then double it so 6 + 6 = 12
You can skip count 3,6,9,12.

Place your set of tiles on the graph paper to fill the upper left hand corner.
Invite students to do the same.
Across the top of the rectangle we can see 3 tiles, this rectangle has 3 columns so write a 3 above the third column
This rectangle goes down 4 rows, so write a 4 on the outside of the grid to indicate.
The rectangle we have created covers how many squares on the paper?
Twelve.

Lift the tile in the bottom left hand corner of the rectangle and print a 12 to remind you.



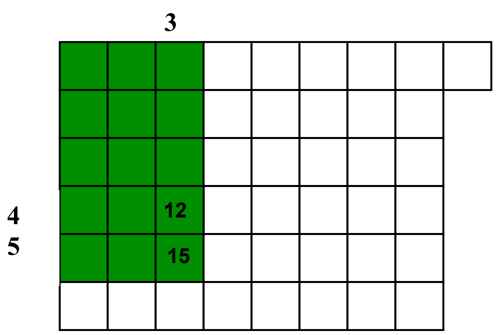
Invite students to slide the tiles off the graph paper and picture the rectangle as it sat on the page.



Can you see the 3 by 4 rectangle?
It covers 12 squares 3 x 4 = 12

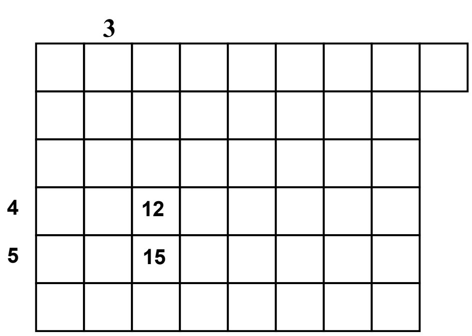
Can you see where the rectangle would fit if it was a 3 by 5?
Build the 3 by 5 rectangle in the grid.
The 3 still tells us we have 3 columns, there are 3 tiles across the rectangle.
Now you need to place a 5 along the outside edge because this rectangle goes down 5.

How many squares does this rectangle cover? (15)
How did you decide?
(I added 3 more to the 12, I counted 3,6,9,12,15, I thought 3 and 3 is 6, 6 and 6 is 12 and 3 is 15, I thought 12,13,14,15)
So the new area is 15, write 15 under the last tile.



You can describe the new area as
(3 x 4) + 3 = 15
Or as 3 x 5 = 15
So (3 x 4) + 3 = 3 x 5

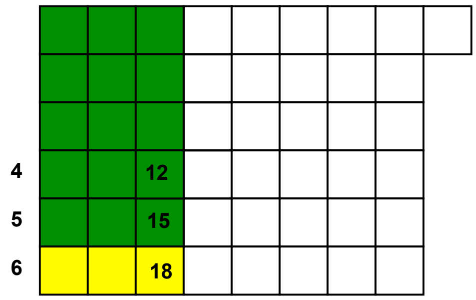
Remove the tiles from the grid and study the rectangle. Can you see the 3 by 5 rectangle? Can you see the 3 by 4 inside it? How are the 2 rectangles related?
The 3 x 5 has 3 more tiles, the 3 x 5 has another row of 3
The 3 x 4 has 3 less tiles than the 3 x 5
So 3 x 4 + 3 = 15.



Can you still see 3 x 4 = 12?
(3 x 4) + 3 = 15
and 3 x 5 = 15

Now try to visualize the 3 by 6 How many square tiles will it cover? How do you know?
Build it on the grid. Can you see (3 x 5) + 3 = 18
I use the brackets so that you will realize that we multiplied first then we added.

Write an 18 to show the new area.
Put 6 at the side.
Ask students how they could name
the new area



> You can describe the new area
> as 3 x 6 = 18 or
> (3 x 5) + 3 = 18
> or (3 x 4) + 3 + 3 = 18
> or (3 x 4) + 6 = 18
> How are these all related?

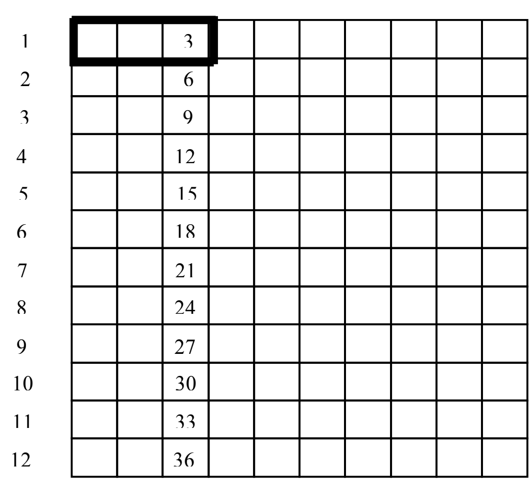
Continue having students build the grid out to 3 x 12.

Now ask students to look at the list of areas and consider what they see.
Do you see a pattern?Can you work backwards in the pattern to fill the missing areas?

Invite students to visualize the rectangles as they work backwards:
Look closely, can you see the 3 x 3 rectangle, it is actually a square.
Can you see the area of 9? Write in 9.

Do you <u>see</u> the 3 x 2 rectangle?
What is its area?
Write in the 6

Students often struggle to make sense of 3 x 1 as a rectangle. Seeing it in the matrix can be very helpful.
Invite students to outline the 3 x 1 before filling in the area.



Now take a few minutes and with a partner or alone record the patterns you can see in these numbers.

What stays the same each time?
3 columns, three in each row, 3 across the top, the width

What changes each time?
The number of rows, the total number of tiles, the number of squares, the area, the length of the rectangle, one side gets longer

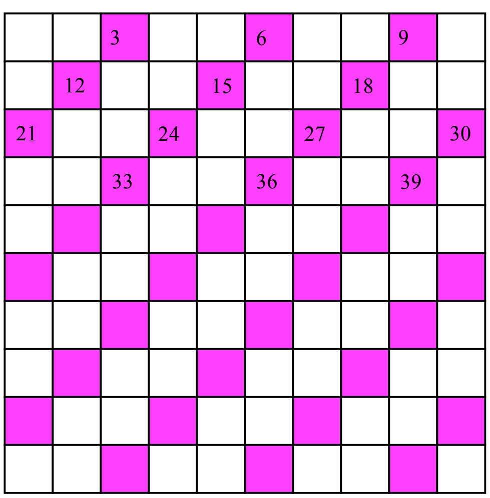
While students are looking at the patterns record the list vertically on the board or on a chart paper.

3
6
9
12
15
18
21
24
27
30
33
36
39
41
44
47

Encourage students to continue the pattern

Patterns students notice:
They jump by threes, you add 3 each time, there are 3 ones in the tens place (three teen numbers 12,15,18), then three twos in the tens position because there are 3 twenty numbers (21,24,27) and then there are four 3's in the tens place because there are four thirty numbers (30,33,36,39).
Can you predict where the pattern might go? How many forty numbers do you predict? How about how many numbers in the fifties? Do you think there will be a four numbers in the sixties? Where will the tens digit repeat four times again?
If students continue the pattern they notice that the ones column repeats 3,6,9,2,5,8,1,4,7,0 They are out of order but all the digits are represented o to 9.

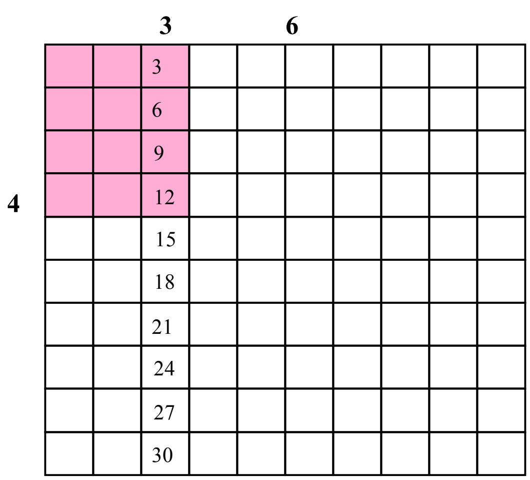
The digits in each number add to a multiple of 3, 6 or 9. So 12 is 1 + 2, 21 is 2 + 1, 27 is 2 + 7, If students do not notice this pattern you might want to suggest it to them and see if they agree. So will 81 be in this list? How about 91?



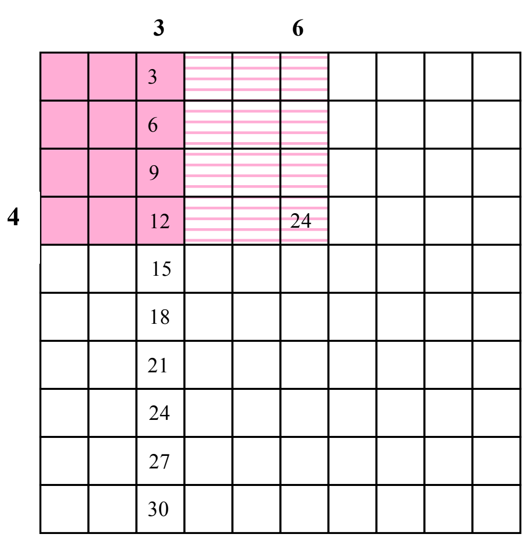
If students transfer the pattern to a hundred chart a different set of patterns emerges.
Now what patterns do you notice?
What can you predict for the 101 to 200 chart?

Noting similarities and differences is how the brain makes sense. This inductive strategy is one of the most powerful instructional strategies and has the potential to reach the greatest number of students. (Marzano, 2001; Bransford, 1999). Students need to investigate all the links between area, arrays, repeated addition and multiplication. Some will hook to the number patterns, some to the spatial arrays. The more students discuss, explore and connect ideas about these shapes, the more likely they will be to internalize the information in connected networks of meaningful and easily retrieved information.

Use the same format to build the six times table with arrays. It will be inserted onto the matrix with the threes.

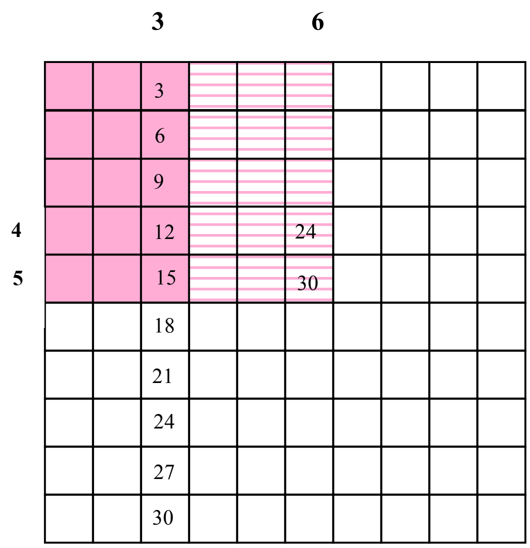


Place the 3 by 4 array back in the matrix.
Can you use it to visualize the 6 by 4?
Can you see where it will sit?
How does it compare to the 3 by 4?
Can you describe the 6 x 4 in terms of the 3 x 4? In other word can you use the 3 x 4 to explain the 6 x 4?
3 x 4 + 3 x 4 = 6 x 4
2 times the 3 x 4 = 6 x 4 or 2 (3x4) = 6 x 4
Double the 3 x 4 = 6 x 4



Can you see the area that the 6 x 4 covers?
It is twice the 3 x 4 so the area is 24.
(3 x 4)+ (3 x 4) = 6 x 4
or 2 x (3 x 4) = 6 x 4

Can you see it?



Let’s use the 3 x 5 to make the 6 x 5.
Before we start, can you see it?
Visualize in your mind.

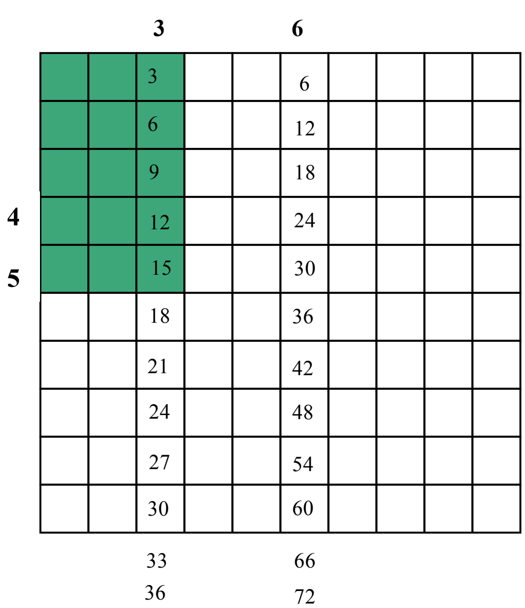
Place the 3 x 5 on the matrix and then use a slide or flip to create the 6 x 5.

How can we name this new area?
6 x 5 = 30
(3 x 5) + (3 x 5) = 30
(15) + (15) = 30
(3 x 5) + (3 x 5) = 6 x 5
2 x (3 x 5) = 6 x 5
2 x (3 x 5) = 30

Have students continue to use the threes to build the sixes.
Once they have finished to 6 x 12, step back and look at the pattern.
Can you work backwards to fill in the 6 x 3 ?
(3 x 3) + (3 x 3) = 6 x 3 or (6 x 4) – 3 = 6 x 3

Encourage students to visualize the rectangles, before they fill in numbers.

As with the three times table, invite students to record the list of multiples of 6 and look for patterns.
They increase by 6 each time, they add to 6,3,9 instead of 3,6,9, they are all even, the ones run in a pattern that goes 0,6,2,8,4,0,6,2,8,4, in the tens there is 0 and another 0 (for the six) then two tens, then one twenty, then two thirties, then two forties, then one fifty, then two sixties, and I predict two seventies, then one eighty…..



Can you see the 3 times tables in the sixes?

What do you notice when you compare the three times table to the six times table?

Which multiples do they share?

					6				
	12						18		
			24						30
					36				
	42						48		

If 3 x 8 = 24 then 6 x 8 = ?

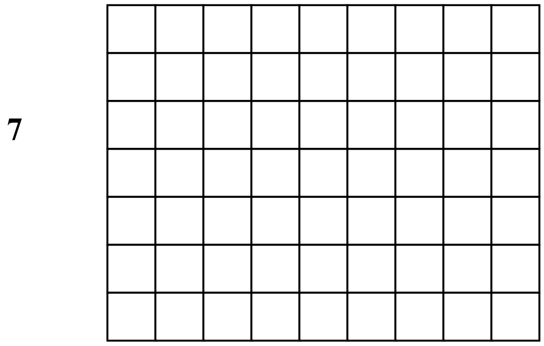
If 4 x 4 = 16 then 8 x 4 = ?

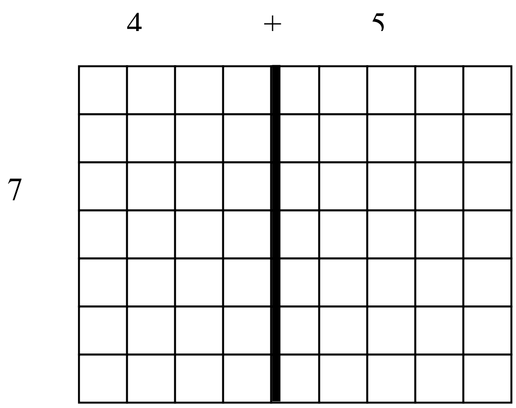
If 6 x 7 = 42 then 6 x 8 = 42 + 6

If 8 x 8 = 64 then 8 x 9 is the same as 64 + ?

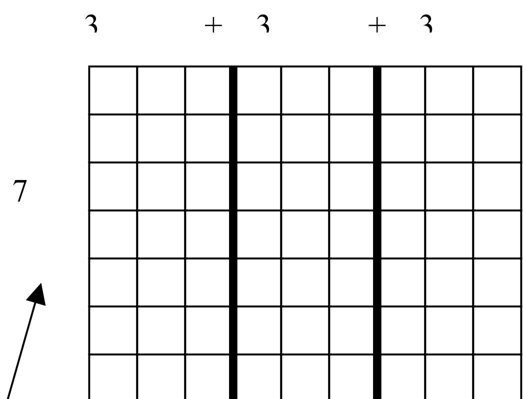
If 6 x 10 = 60 then 6 x 9 = ?

If 12 x 12 = 144 then 12 x 11 = 144 - ?





$(4 + 5) \times 7 = 9 \times 7$
$(4 \times 7) + (5 \times 7) = 9 \times 7$



$9\ (2 + 5) = 9 \times 7$
$(9 \times 2) + (9 \times 5)$

$(3 \times 7)+(3 \times 7)+ (3 \times 7) = 9 \times 7$
$(3 + 3 + 3) \times 7 = 9 \times 7$