

The Importance of Equal Sign Understanding

in the Middle Grades

Eric J. Knuth, Martha W. Alibali, Shanta Hattikudur,
Nicole M. McNeil, and Ana C. Stephens

The equal sign is perhaps the most prevalent symbol in school mathematics, and developing an understanding of it has typically been considered mathematically straightforward. In fact, after its initial introduction during students' early elementary school education, little, if any, instructional time is explicitly spent on the concept in the later grades. Yet research suggests that many students at all grade levels have not developed adequate understandings of the meaning of the equal sign (Baroody and Ginsburg 1983; Behr, Erlwanger, and Nichols 1980; Falkner, Levi, and Carpenter 1999; Kieran 1981; Knuth et al. 2006). Such



Eric Knuth, knuth@education.wisc.edu, is a mathematics education professor at the University of Wisconsin—Madison. He is interested in the development and cultivation of students' mathematical



reasoning. **Martha Alibali**, mwalibali@wisc.edu, is professor of psychology and educational psychology at the University of Wisconsin—Madison. She studies the development of mathematical knowledge and communication about mathematical ideas. **Shanta Hattikudur**,

hattikudur@wisc.edu, is a third-year graduate student in psychology at the University of Wisconsin—Madison. She is studying how linking concepts can ease students' transition from arithmetic to algebraic thinking. **Nicole M. McNeil**, nmcneil@nd.edu, is an assistant professor of psychology and a Fellow of the Institute for Educational Initiatives at the University of Notre Dame. She studies the development of mathematical reasoning and symbolic understanding. **Ana Stephens**, acstephens@wisc.edu, is an assistant researcher at the Wisconsin Center for Education Research at the University of Wisconsin—Madison. She is interested in students' algebraic thinking and mathematics teacher education.

findings are troubling with respect to students' preparation for algebra, especially given Carpenter, Franke, and Levi's (2003) contention that a "limited conception of what the equal sign means is one of the major stumbling blocks in learning algebra. Virtually all manipulations on equations require understanding that the equal sign represents a relation" (p. 22).

So what do middle school students think the equal sign means? What might account for shortcomings in the way that many students think about this symbol? Does it matter if students develop a sophisticated understanding of the equal sign? As teachers, what can we do to help students develop their understandings of the equal sign? In the remainder of this article, we address these questions as we share results from our research.

STUDENTS' EQUAL SIGN UNDERSTANDINGS

Middle school students (grades 6–8) were asked to provide a definition for the equal sign symbol (see **fig. 1**). This question required students to name the equal sign symbol (first prompt); provide a statement regarding the symbol's meaning (second prompt); and then, if possible, provide a statement regarding an alternative meaning (third prompt). The rationale for the first prompt was to preempt students from using the name of the symbol in their response to the second prompt (e.g., "the symbol means equal"). The rationale for the third prompt was based on our previous work, in which we found that students often offer more than one interpretation when given the opportunity.

Before reading on, take a minute to think about how your own students might respond to this question. (Better yet, think about how your students might respond, present the question to your students, and compare your predictions with their actual responses.)

Fig. 1 Interpreting the equal sign

The following questions are about this statement:

$$3 + 4 = 7$$

↑

- The arrow above points to a symbol. What is the name of the symbol?
- What does the symbol mean?
- Can the symbol mean anything else? If yes, please explain.

We classified student responses to parts (b) and (c) into four categories: *relational*, *operational*, *other*, or *no response/don't know*. A response was categorized as *relational* if the student expressed the general idea that the equal sign represents an equivalence relation between two quantities—the definition we would like to see from students. The following student responses are representative of those categorized as relational:

- "It means that what is to the left and right of the sign mean the same thing." (grade 6 student)
- "The same as, same value." (grade 7 student)
- "The left side of the equals sign and the right side of the equals sign are the same value." (grade 8 student)

A response was categorized as *operational* if the student expressed the general idea that the equal sign means "add the numbers" or "the answer." The following student responses are representative of those categorized as operational:

- "What the sum of the two numbers are." (grade 6 student)
- "A sign connecting the answer to the problem." (grade 7 student)
- "The total." (grade 8 student)
- "How much the numbers added together equal." (grade 8 student)

Table 1 Percent of students at each grade level who provided each type of equal sign definition as their best definition ($n = 375$)

| Best Definition | Grade 6 | Grade 7 | Grade 8 |
|----------------------------|---------|---------|---------|
| Relational | 29 | 36 | 46 |
| Operational | 58 | 52 | 45 |
| Other | 7 | 9 | 8 |
| No response/ don't know | 6 | 3 | 1 |

The *other* category included definitions such as “it means equals” or “it means equal to” as well as direct translations of the problem statement, such as “3 plus 4 equals 7.” In addition to classifying responses to parts (b) and (c) separately, students were also assigned an overall category indicating their “best” definition. Many students provided two definitions, sometimes one *relational* and one *operational*; in such cases, the responses were assigned an overall category of *relational*.

Table 1 presents the distribution of equal sign definitions as a function of grade. (Recall that in cases where students provided two definitions, they were assigned an overall category for their best definition; thus, **table 1** displays students’ best definitions. There were fewer students who *only* provided a definition that was categorized as relational—11 percent in grade 6, 16 percent in grade 7, and 24 percent in grade 8.) How do these data compare with your predictions about student responses? As seen in the table, the majority of students provided definitions that were categorized as operational, whereas substantially fewer students provided definitions categorized as relational. Although we do see improvement in the definitions categorized as relational as students progress through middle school, increasing from 29 percent in grade 6 to 46 percent in grade 8, such improvement is relatively modest. Overall, the results are somewhat discouraging—fewer than

half the students in each grade demonstrated a relational view of the equal sign. Perhaps not surprising, given the lack of attention the equal sign receives in middle school curricula and instruction, an understanding of the equal sign does not seem to improve significantly as students progress through middle school.

WHAT MIGHT ACCOUNT FOR THE PREDOMINANCE OF AN OPERATIONAL VIEW OF THE EQUAL SIGN?

Researchers have argued that the operational view of the equal sign is largely a by-product of students’ experiences with the symbol in elementary school mathematics (e.g., Baroody and Ginsburg 1983; Carpenter et al. 2003; McNeil and Alibali 2005). During elementary school, students typically encounter the equal sign in number sentences that have operations on the left side of the equal sign and an answer blank on the right side (e.g., $5 + 2 = \underline{\quad}$, $11 - 4 + 1 = \underline{\quad}$). In solving such “operations equal answer” equations correctly, it is not really necessary for students to think about the equal sign as a symbol of equivalence; rather, students need only perform the calculations on the left side of the equal sign to get an answer. As a result, students associate the equal sign with the arithmetic operations performed to get a final answer. Unfortunately, this operational view of the equal sign is often reinforced

year after year as students gain more and more experience with traditional arithmetic equations during their elementary school years.

Because an operational view of the equal sign is well established by the time students reach middle school, it may be far from trivial for students to acquire a relational view of the equal sign during middle school. Further compounding this problem is the fact that very little attention is paid to the symbol in middle school curricular materials. Our analysis of four popular middle school mathematics textbooks revealed that equations are frequently presented in an “operations on the left side of the equal sign” format and are rarely presented in an “operations on both sides of the equal sign” format (for more detail about the textbook study, see McNeil et al. 2006). This pattern of exposure may actually condition students to favor less sophisticated interpretations of the equal sign and, as we shall see next, such interpretations may have important consequences for students in learning algebra.

DOES IT REALLY MATTER IF STUDENTS VIEW THE EQUAL SIGN RELATIONALLY?

A natural question to ask is whether it really makes a difference when learning mathematics (and algebra, in particular) if students view the equal sign relationally. To answer this question, we asked students to respond to items that are more typical of what they might see in an algebra class, and then we examined their success in solving these items as well as the strategies they used as a function of their views of the equal sign. One set of items required students to determine solutions to linear equations (see **fig. 2**), whereas another set of items required students to make judgments regarding the solutions to equivalent equations (see **fig. 3**).

Fig. 2 Equation-solving items

What value of m will make the following number sentences true?

(a) $4m + 10 = 70$

(b) $3m + 7 = 25$

The equation-solving items required students to determine the solution to typical beginning algebra equations. We used the prompt “What value of m will make the following number sentences true?” for two reasons. First, we wanted to avoid running into problems associated with students misunderstanding the more typical (in terms of beginning algebra) prompt, “Solve the equation.” Many students in middle school, particularly sixth- and seventh-grade students, would likely be unfamiliar with such a prompt. Second, we felt that the prompt would likely be more consistent with language often used in elementary school (i.e., “What value will make the number sentence $8 + \square = 12$ true?”).

The equivalent equations items required students to determine whether the missing value for one equation was the same as the missing value for a second (equivalent) equation. In both cases, we expected that students who viewed the equal sign as representing an equivalence relation between quantities (a relational view of the equal sign) would recognize that the operations performed on the second equation of each pair preserved the quantitative relationship expressed in the first equation of each pair. It is also worth noting that the operations applied to the second (equivalent) equation in both items are atypical in that they are not operations that would normally be applied in the process of solving the first equation (e.g., subtracting 15 from both sides of the equation). We decided to use atypical operations to deter students from simply recognizing

Fig. 3 Equivalent equations items

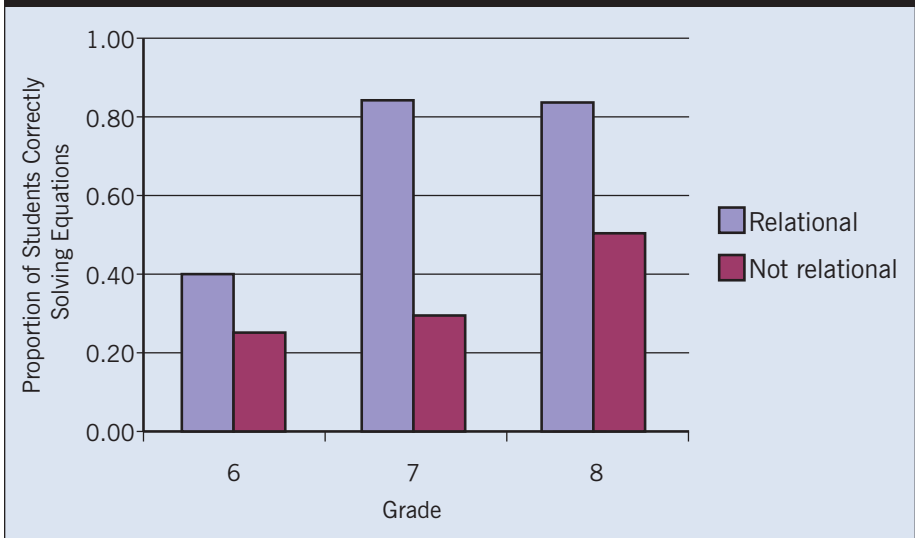
(a) Is the number that goes in the \square the same number in the following two equations? Explain your reasoning.

$$2 \times \square + 15 = 31$$

$$2 \times \square + 15 - 9 = 31 - 9$$

(b) In the equation $\square + 18 = 35$, the number that goes in the \square is 17. Can you use this fact to help you figure out what number goes in the \square in the equation $\square + 18 + 27 = 35 + 27$? Explain your reasoning.

Fig. 4 Proportion of sixth-, seventh-, and eighth-grade students in each equal sign understanding category that solved the linear equations correctly ($n = 177$)



that the correct procedure was applied and then relying on that recognition to respond that the solutions to the pair of equations are the same.

So how did students perform? **Figure 4** shows that students who exhibited a relational view of the equal sign were more likely than students who did not to solve the linear equation items (**fig. 2**) correctly. In other words, at each grade level, a greater proportion of students who provided a relational definition of the equal sign solved the equations correctly. This finding is noteworthy in that it suggests that even students having no experience with formal algebra (sixth- and seventh-grade students, in particular) have a better understanding of how to solve equations when they hold a relational view of the equal sign.

Similar results were found for the equivalent equations items as well. For the first equivalent equations item (**fig. 3a**), we see in **figure 5** that at each grade level, students who provided a relational definition of the equal sign were more likely than their peers who did not to correctly respond that the two equations have the same solution.

We can also see further evidence for the importance of viewing the equal sign relationally if we consider the strategies that students used in arriving at their answers to both equivalent equations items. Students employed a variety of strategies that led to correct responses. For example, some students solved each equation and then compared the two solutions; others used a variation of this strategy in which they solved the first equation

Fig. 5 Proportion of sixth-, seventh-, and eighth-grade students in each equal sign understanding category that solved the equivalent equations item correctly ($n = 127$)

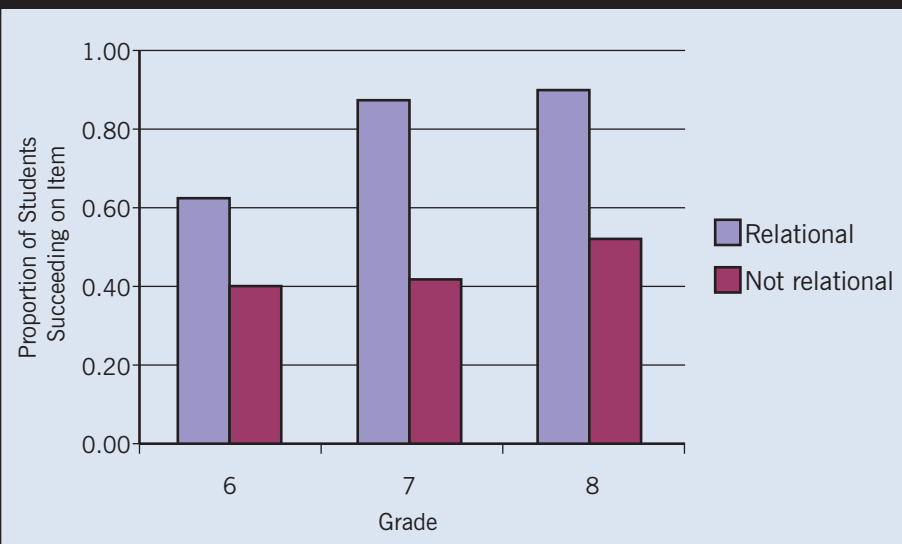
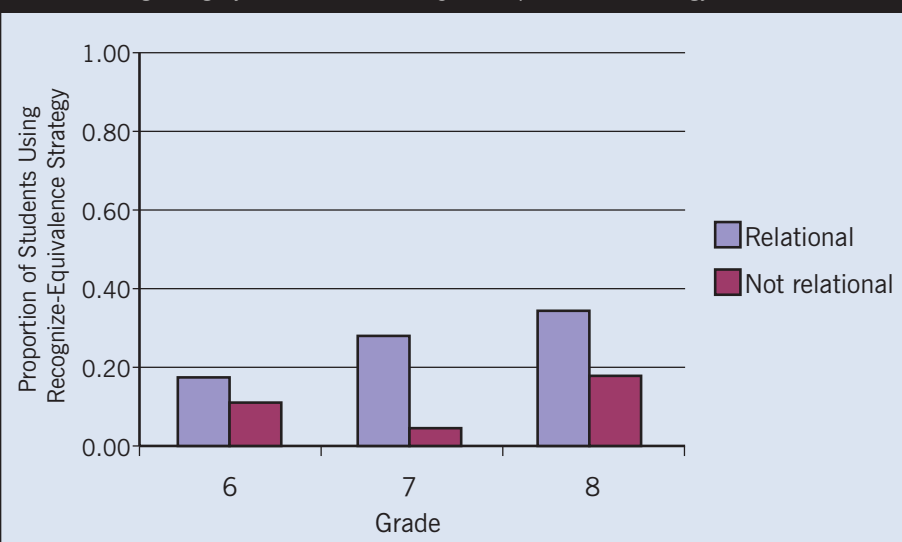


Fig. 6 Proportion of sixth-, seventh-, and eighth-grade students in each equal sign understanding category that used the *recognize-equivalence* strategy ($n = 249$)



and then substituted that value into the second equation. However, the most efficient strategy and, arguably, the most mathematically sophisticated strategy is to recognize that the operations performed on the second equation of each pair preserves the equivalence relation expressed in the first equation (i.e., doing the same thing to both sides of the equation does not alter the solution to the equation). The following student responses, the first to the first item and the second to

the second item, are representative of the *recognize-equivalence* strategy:

Yes, if both sides of the expressions are multiplied, subtracted, added, divided, or anything else, by the same amount, the two expressions are still equivalent. (grade 8 student)

Yes it can help by saying that you can add 27 to either side of the equal sign and the answer would still be the same. (grade 7 student)

One may wonder whether such responses reflect an understanding of the equivalence relationship expressed by an equation or the memorization of a rule (i.e., “same thing to both sides”). When asked to further explain such responses in interviews, students would often provide computational examples to illustrate that they do indeed understand the equivalence relationship expressed by an equation. Overall, **figure 6** shows that at each grade level, the *recognize-equivalence* strategy was more likely to be used by those students who provided a relational definition of the equal sign.

Looking across all the items, it is clear that those students who view the equal sign relationally outperformed their peers who do not hold a relational view on equation-solving and equivalent equations problems. Moreover, they were more likely to use a more mathematically sophisticated strategy in solving the problems. Thus, in answering the question posed at the beginning of this section, we see that it *does* matter if students view the equal sign relationally.

HELPING STUDENTS TO DEVELOP A RELATIONAL VIEW OF THE EQUAL SIGN

We found a potentially important implication from our research: Helping students acquire a view of the equal sign as a symbol that represents an equivalence relation between two quantities may, in turn, help prepare them for success in algebra (and beyond). (For further information about the research reported in this article or for copies of the research articles, visit labweb.education.wisc.edu/knuth/taar/.) Yet much work is to be done if we consider that fewer than 50 percent of the middle school students in our study demonstrated a relational view of the equal sign (and similar results have been found in other studies). Thus, we argue that there

is a clear need for continued attention to be given to the notion of equality in the middle school grades.

In our professional development work with middle school teachers, for example, we encourage teachers to look for opportunities within their existing classroom practices to engage students in conversations about the equal sign as well as to create such opportunities intentionally. As an example of a “naturally” occurring opportunity, most mathematics teachers have likely witnessed the equality “strings” that students often produce (e.g., $3 + 5 = 8 + 2 = 10 + 5 = 15$); these equality strings provide an excellent opportunity to discuss with students the meaning of the equal sign and its proper use. To create opportunities intentionally, teachers might provide students with arithmetic (or algebraic) equations to solve in which numbers and operations (or symbolic expressions and operations) appear on both sides of the equal sign. Such equation formats may help promote more appropriate interpretations and uses of the equal sign. Teachers could also provide students with problems such as the equivalent equations items discussed in this article or their arithmetic counterparts (e.g., Is $3 + 5 = 6 + 2$ the “same as” $3 + 5 - 4 = 6 + 2 - 4$?). Discussing with students the different approaches to solving such problems can help foster more appropriate interpretations of the equal sign. Opportunities such as these can be implemented so that they take little time away from an already crowded curriculum (e.g., as warm-ups), yet still allow teachers to underscore the relational view of the equal sign.

Improving students’ understanding of the equal sign, and thus their preparation for algebra, may require changes in teachers’ instructional practices as well as changes in elementary and middle school mathematics curricula. The notion of equality is surprisingly

complex, is often difficult for students to comprehend (RAND 2003), and “should be developed throughout the curriculum” (NCTM 2000, p. 39). We encourage teachers to both recognize and create opportunities to foster students’ thinking about the equal sign. Such efforts may provide rich dividends as more students are better prepared for success in algebra.

REFERENCES

- Baroody, Arthur, and Herbert Ginsburg. “The Effects of Instruction on Children’s Understanding of the ‘Equals’ Sign.” *Elementary School Journal* 84 (1983): 199–212.
- Behr, Merlyn, Stanley Erlwanger, and Eugene Nichols. “How Children View the Equals Sign.” *Mathematics Teaching* 92 (1980): 13–15.
- Carpenter, Thomas, Megan Franke, and Linda Levi. *Thinking Mathematically: Integrating Arithmetic and Algebra in the Elementary School*. Portsmouth, NH: Heinemann, 2003.
- Falkner, Karen, Linda Levi, and Thomas Carpenter. “Children’s Understanding of Equality: A Foundation for Algebra.” *Teaching Children Mathematics* 6 (1999): 56–60.
- Kieran, Carolyn. “Concepts Associated with the Equality Symbol.” *Educational Studies in Mathematics* 12 (1981): 317–26.
- Knuth, Eric, Ana Stephens, Nicole McNeil, and Martha Alibali. “Does Understanding the Equal Sign Matter? Evidence from Solving Equations.” *Journal for Research in Mathematics Education* 36 (July 2006): 297–312.
- McNeil, Nicole, and Martha Alibali. “Knowledge Change as a Function of Mathematics Experience: All Contexts Are Not Created Equal.” *Journal of Cognition and Development* 6 (2005): 285–306.
- McNeil, Nicole, Laura Grandau, Eric Knuth, Martha Alibali, Ana Stephens, Shanta Hattikudur, and Daniel Krill. “Middle-School Students’ Understanding of the Equal Sign: The Books

- They Read Can’t Help.” *Cognition and Instruction* 24 (March 2006): 367–85.
- National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000.
- RAND Mathematics Study Panel. *Mathematical Proficiency for All Students: Toward a Strategic Research and Development Program in Mathematics Education*. Santa Monica, CA: RAND, 2003. ●