Final Exam Review Part 2

Exponential & Logarithmic Functions and Equations

Polynomial & Sinusoidal Functions

Rational Expressions and Equations

Exponential Functions

- *To describe, orally and in written form, the characteristics of an exponential function by analyzing its graph*
- *To describe, orally and in written form, the characteristics of an exponential function by analyzing its equation*
- *To be able to match equations in a given set to their corresponding graphs*
- *To determine the solution of an exponential equation in which the bases are powers of one another*
- *To determine the solution of an exponential equation in which the bases are not powers of one another*
- *To graph data, and determine the exponential function that best approximates the data*
- *To interpret the graph of an exponential function that models a situation, and explain the reasoning*
- *To solve, using technology, a contextual problem that involves data that is best represented by graphs of exponential functions and explain the reasoning*
- *To solve problems that involve the application of exponential equations to loans, mortgages and investments*
- *To graph data, and determine the exponential function that best approximates the data*

An exponential function is a function whose equation is of the form

$y = ab^x$, where $a \neq 0, b > 0, x \in R$

Characteristics of the Graph of the Exponential Function:

Function

- the y-intercept is $(0, a)$
- there is no x-intercept
- the x-axis is a horizontal asymptote
- the domain is real
- the range is $\{y > 0, y \in R\}$
- For a > 0:
  - when b > 1, the function represents a Growth function
  - when 0 < b < 1, the function represents a decay function
- The value of b affects the steepness of the graph as x increases:
  - when b > 1, the curve rises sharply as b increases
  - when 0 < b < 1, the curve falls sharply as b decreases

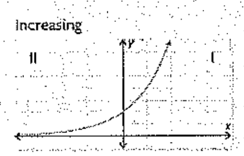
Need to Know

- There are two different shapes of the graphs of an exponential function of the form $f(x) = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$:

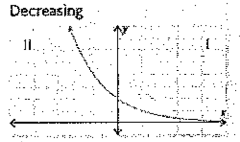
  - Case 1: An increasing function; the curve extends from quadrant II to quadrant I.

  - Case 2: A decreasing function; the curve extends from quadrant II to quadrant I.

Increasing

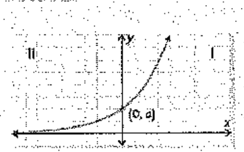


Decreasing

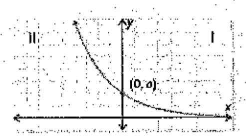


Need to Know

- An exponential function is an increasing function if $a > 0$ and $b > 1$.
- An exponential function is a decreasing function if $a > 0$ and $0 < b < 1$.
- Changing the parameters a and b in exponential functions of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, does not change the number of x-intercepts, the end behaviour, the domain, or the range of the function. These characteristics are identical in all exponential functions of this form.



$a > 0, b > 1$



$a > 0, 0 < b < 1$

Ex. Predict the number of x-intercepts, the y-intercept, the end behavior, the domain, and the range of the following functions:

a. $f(x) = 2(5)^x$

x-int = none
y-int = 2
dom: $x \in \mathbb{R}$
Range $\{y > 0, y \in \mathbb{R}\}$

b. $y = e^x$

x-int = none
y-int = 1
dom: $x \in \mathbb{R}$
Range $\{y > 0, y \in \mathbb{R}\}$

e also equals 2.718

same

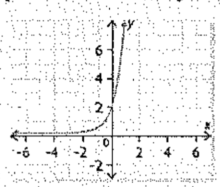
c. $f(x) = 8\left(\frac{3}{4}\right)^x$

y-int = 8

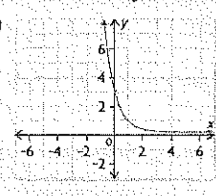
same

Which exponential function matches each graph below? Provide your reasoning.

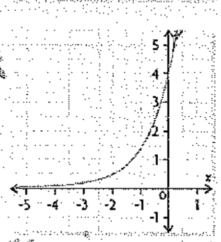
i) $y = 3(0.2)^x$ ii) $y = 4(3)^x$ iii) $y = 4(0.5)^x$ iv) $y = 2(4)^x$

a)


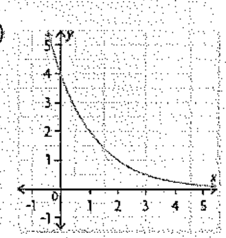
$y = 2(4)^x$

c)


$y = 3(0.2)^x$
y-int decreasing

b)


$y = 4(3)^x$
y-int increasing

d)


$y = 4(0.5)^x$

Logarithmic Functions

Learning Outcomes:

- *To describe, orally and in written form, the characteristics of a logarithmic function by analyzing its graph*
- *To describe, orally and in written form, the characteristics of a logarithmic function by analyzing its equation*
- *To match equations in a given set to their corresponding graphs*
- *To express a logarithmic equation as an exponential equation and vice versa*
- *To determine the value of a logarithmic expression without technology*
- *To solve problems that involve logarithmic scales, such as Richter Scale and the pH scale*
- *To develop the laws of logarithms, using numeric examples and exponent laws*
- *To determine an equivalent expression for a logarithmic expression by applying the laws of logarithms*
- *To determine the approximate value of a logarithmic expression, such as $\log_2 9$, with technology*
- *To determine the solution of an exponential equation in which the bases are not powers of one another*
- *To graph data, and determine the logarithmic function that best approximates the data*
- *To interpret the graph of a logarithmic function that models a situation, and explain the reasoning*
- *To solve, using technology, a contextual problem that involves data that is best represented by graphs of logarithmic functions, and explain the reasoning*

Logarithmic Function: A function of the form $y = a\log_b x$, where $b > 0$, $b \neq 1$, and $a \neq 0$, and a and b are real numbers.

The function $y = \log_{10} x$ is equivalent to $x = 10^y$, so a logarithm is an exponent. The meaning of $\log_{10} x$ is "the exponent that must be applied to base 10 to get the value of x." For example, $\log_{10} 100 = 2$.

A logarithm with base e is called the natural logarithm and is written as $\ln x$. The functions $y = \log_e x, y = \ln x, and\ x = e^y$ are equivalent.

Ex. Predict the x-intercepts, number of y-intercepts, the end behavior, the domain and range of the following functions:

a. $y = -5\log x$

x int = 1
y int = 0
end: Q4 - Q1
d: $x > 0, x \in \mathbb{R}$
r: $y \in \mathbb{R}$

b. $y = 12\ln x$

x-int = 1
y-int = none
end behavior: Q4 to Q1
d: $x > 0, x \in \mathbb{R}$
R: $y \in \mathbb{R}$

- A logarithmic function has the form $f(x) = a\log_b x$, where $b > 0$, $b \neq 1$, and $a \neq 0$, and a and b are real numbers.
- All logarithmic functions of the form $f(x) = a\log x$ and $f(x) = a\ln x$ have these characteristics:

x-Intercept	1
Number of y-Intercepts	0
End Behaviour	The curve extends from quadrant IV to quadrant I or quadrant I to quadrant IV.
Domain	$\{x \mid x > 0, x \in R\}$
Range	$\{y \mid y \in R\}$

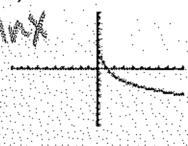
- All logarithmic functions of the form $f(x) = a\log x$ and $f(x) = a\ln x$ have these unique characteristics:
  - If $a > 0$, the function increases.
  - If $a < 0$, the function decreases.

Ex. Which function matches each graph below? Provide your reasoning.

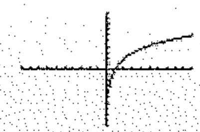
i) $y = 5(2)^x$ ii) $y = 2(0.1)^x$ iii) $y = 6\log x$ iv) $y = -2\ln x$

a)

$y = -2\ln x$

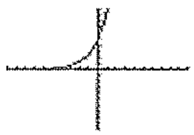


b)



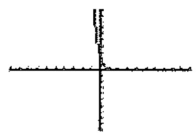
$y = 6\log x$ ← logarithmic

c)



$y = 5(2)^x$

d)

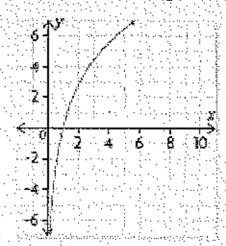


$y = 2(0.1)^x$

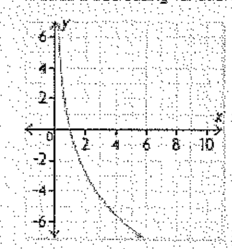
← exponential

- The graph of a logarithmic function of the form $f(x) = a\log x$ or $f(x) = a\ln x$ will look like one of the following cases:

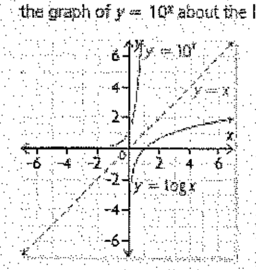
Case 1: an increasing function, where $a > 0$



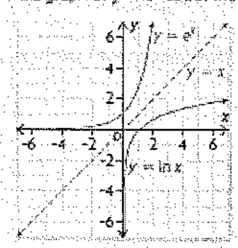
Case 2: a decreasing function, where $a < 0$



- The graph of $y = \log x$ is a reflection of the graph of $y = 10^x$ about the line $y = x$.



- The graph of $y = \ln x$ is a reflection of the graph of $y = e^x$ about the line $y = x$.



Key Ideas

- The logarithmic function
  $$y = \log_b x$$
  is equivalent to the following exponential function:
  $$x = b^y$$
- A logarithm is an exponent. The expression $\log_b x$ means "the exponent that must be applied to base b to give the value of x." For example:
  $$\log_2 8 = 3$$
  since
  $$2^3 = 8$$
- The value of a logarithm can be determined in one of the following ways:
  - Set the logarithmic expression equal to y, and write the equivalent exponential form. Then determine the exponent to which the base must be raised to get the required number.
  - If the base of the logarithm is 10 or e, you can use a scientific or graphing calculator.
- The common logarithmic function
  $$y = \log x$$
  is equivalent to the following exponential function:
  $$x = 10^y$$
- The natural logarithmic function
  $$y = \ln x$$
  is equivalent to the following exponential function:
  $$x = e^y$$
- The logarithm of a negative number does not exist, because a negative number cannot be written as a power of a positive base.
- Many real-life situations have values that vary greatly. A logarithmic scale with powers of 10 can be used to make comparisons between large and small values more manageable. Three examples of logarithmic scales are the Richter scale (used to measure the magnitude of an earthquake), the pH scale (used to measure the acidity of a solution), and the decibel scale (used to measure sound level).

Changing from Logarithmic form to Exponential Form:

$$\log_b n = x \Leftrightarrow b^x = n$$

a. $\log_7 x = 4$ b. $\log_{10} 1000 = 3$ c. $\log_x a = b^d$

$7^4 = x$ $10^3 = 1000$ $x^{b^d} = a$

Changing from Exponential to Logarithmic Form:

a. $4^3 = 64$ b. $2^{-3} = \frac{1}{8}$ c. $a = (2x+4)^{-1}$

$\log_4 64 = 3$ $\log_{\frac{1}{8}} 2 = -3$ $\log_a (2x+4) = -1$

Ex. Determine the value of y in the following exponential equations, then verify your answer:

a. $32 = 10^y$

$y = \log_{10} 32$

$=$

b. $25 = e^y$

$y = \ln 25$

Change of base identity: Used to solve for individual logarithms where the same base can't be found.

$$\log_b c = \frac{\log_a c}{\log_a b}$$

Ex. Evaluate:

a. $\log_4 1024$

$$\frac{\log 1024}{\log 4} = 5$$

b. $\log_4 450$

$$\frac{\log 450}{\log 4} = 4.4$$

c. $3\log_7 512$

$$\frac{3\log 512}{\log 7} = 9.6$$

Ex. Solve for the given variable or evaluate:

a. $y = \log_5 \sqrt{125}$ b. $\log_4 64$ c. $\log_2\left(\frac{1}{32}\right)$

$$\frac{\log \sqrt{125}}{\log 5} = 1.5 \qquad \frac{\log 64}{\log 4} = 3 \qquad \frac{\log \frac{1}{32}}{\log 2} = -5$$

OR

$$2^x = \frac{1}{32}$$

$$2^x = \frac{1}{2^5}$$

$$2^x = 2^{-5}$$

so

$$x = -5$$

Laws of Logarithms

Since logarithms are exponents, the laws of logarithms are related to the laws of powers

Product Law of Logarithms

$$log_c MN = log_c M + log_c N$$

Quotient Law of Logarithms

$$log_c \frac{M}{N} = log_c M - log_c N$$

Power Law of Logarithms

$$log_c M^P = P log_c M$$

Exponent Laws:

Product Law: $x^m x^n = x^{m+n}$

Quotient Law: $x^m \div x^n = x^{m-n}$

Power of a Power: $(x^m)^n = x^{mn}$

Power of a Product: $(xy)^m = x^m y^m$

Power of a Quotient: $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}, y \neq 0$

Integral Exponent Rule: $x^{-m} = \frac{1}{x^m}, y \neq 0$

Rational Exponents: $x^{\frac{m}{n}} = \sqrt[n]{x^m} \text{ or } \left(\sqrt[n]{x}\right)^m$

Convert $8 \bullet 16^x$ to base 2:

Ex. Simplify and then evaluate each logarithmic expression

a. $log_2 48 - log_2 3$

$\log_2 \frac{48}{3} = \log_2 16 = \boxed{4}$

b. $\log_6 9 + \log_6 8 - \log_6 2$

$\log_6 \frac{9 \cdot 8}{2} = \log_6 36 = \boxed{2}$

c. $2\log 5 + 2\log 2$

$\log 5^2 + \log 2^2 = \log 25 \cdot 4 = \log 100 = \boxed{2}$

d. $2log_3 6 + log_3 0.75$

$\log_3 6^2 \cdot \left(\frac{3}{4}\right) = \log_3 27 = \boxed{3}$

Solving exponential equations

Method 1: change to a common base

- Write both sides of the equation in the same base
- Equate the exponents on both sides of the equation
- Determine the value of the variable

Ex) $13^x = \frac{1}{169}$

$13^x = 13^{-2}$

$x = -2$

$9^{1-x} = 27^{x+2}$

$3^{2(1-x)} = 3^{3(x+2)}$

Method 2: use logarithms

- log both sides of the equations
- use log laws to simplify and isolate the variable

Ex) $5^{3x} = 3^{2x-1}$

$\log 5^{3x} = \log 3^{2x-1}$

$3x\log 5 = (2x-1)\log 3$

$3x\log 5 = 2x\log 3 - \log 3$

$3x\log 5 - 2x\log 3 = -\log 3$

$x(3\log 5 - 2\log 3) = -\log 3$

$x = \frac{-\log 3}{3\log 5 - 2\log 3} = \boxed{-0.418}$

Method 3: use a graphing calculator

- enter each side of the equation as Y1, Y2
- Calculate the intersection point

Ex) $5^{x+2} = 104$

$Y_1 = 5^{(x+2)}$

$Y_2 = 104$

2nd trace 5 Enter Enter Enter

$\boxed{X = 0.886}$

Applications of Exponential & Logarithmic Functions

Compound Interest

Growth/Decay

PH/Richter Scale

Compound Interest: The interest earned on both the original amount that was invested and any interest that has accumulated over time.

The formula which can be used to calculate compound interest is

$$A = P(1+\frac{i}{n})^{nt}$$

$A = P(1+i)^n$ ↑ Formula Sheet

where,
A represents the final amount or future value
P represents the initial amount (principal)
i represents the interest rate
n represents the number of compounding periods

Compounding Period	Number of Times interest is paid	Interest rate per compounding period, i
Daily	365 times per year	$i = \frac{annual\ rate}{365}$
Weekly	52 times per year	$i = \frac{annual\ rate}{52}$
Semi-monthly	24 times per year	$i = \frac{annual\ rate}{24}$
Monthly	12 times per year	$i = \frac{annual\ rate}{12}$
Quarterly	4 times per year	$i = \frac{annual\ rate}{4}$
Semi-annually	2 times per year	$i = \frac{annual\ rate}{2}$
Annually	1 time per year	$i = \frac{annual\ rate}{1}$

↑ x # of years

Ex. Brittany invested \$3000 at a rate of 3.5%/a, compounded semi-annually.

a. How much will her investment be worth at the end of each of the next 4 years?

$3000(1+\frac{0.035}{2})^{(4)(2)} = \$3446.65$

$3000(1.0175)^8$

b. How long, in years will it take her investment to reach \$5000?

solve using logs

$5000 = 3000(1.0175)^n$

$\frac{5000}{3000} = 1.0175^n$

$\log\frac{5}{3} = \log 1.0175^n$

$\log\frac{5}{3} = n(\log 1.0175)$

$\frac{\log 5/3}{\log 1.0175} = n$

$29.445 = n$ # of compounding periods so ÷ by 2 $\boxed{\approx 15 \text{ years}}$

$b = 0.75$
$1 - 0.25$

Ex. A value of a type of robotic technology depreciates 25% per year.

a. Write an exponential function to represent the value of this robotic technology after t years.

$y = a(0.75)^t$ $\quad A_f = A_i(0.75)^t$

b. How many years, to the nearest whole year, would it take for the value of a robot which initially cost \$575 000 to depreciate to a value of \$25 000?

$25000 = 575000(0.75)^t$

[0, 40, 4][0, 100000, 10000] Graph it

$t = 10.9$
$\boxed{= 11 \text{ years}}$

Ex. Jessica borrowed \$7500 from a bank to buy new equipment for her band. The bank is charging an interest rate of 3.6%/a compounded monthly. Jessica's monthly loan payment is \$400.

a. Determine how long, to the nearest month, it will take Jessica to pay off the loan. The loan manager gave Jessica the following equation so she could determine how long it would take to pay off her loan:

$(1.003)^{-n} = 0.94375$

$-n = \frac{\log 0.94375}{\log 1.003} = -19.33 = \boxed{20 \text{ months}}$

[0, 25, 5][0, 1.5, 0.25]

b. How much interest will Jessica pay on the loan?

$20 \times 400 = 8000$
$- 7500$
$\boxed{500}$

Growth/Decay

Formula:
General formula that can be used for solving problems involving doubling time, tripling time, half-life, etc.

$A_t = A_o C^{\frac{t}{p}}$ or as a function $A(t) = A_o C^{\frac{t}{p}}$

where, A_o = initial amount, A_t = amount at time t,

C = a constant, 2 for doubling, 3 for tripling, $\frac{1}{2}$ for half life

t = time, p = period of time for doubling, tripling, halving, etc.

Ex. A patient feeling ill had a sample of bacteria take from her throat. The sample contained 387 cells. Twenty-four hours later the sample contained 8012 cells. Find the doubling period for the bacteria.

$A_t = 8012$, $A_o = 387$, C = 2, t = 24h, p = ?

$8012 = 387(2)^{\frac{24}{p}}$

$\frac{8012}{387} = 2^{\frac{24}{p}}$

$\log 20.703 = \log 2^{\frac{24}{p}}$

$\frac{\log 20.7}{\log 2} = \frac{24}{p}$

$p = \frac{24}{4.37}$

$= 5.49$

$\boxed{= 5.5 \text{ hours doubling period}}$

The Richter scale is logarithmic – a difference in one unit in magnitude corresponds to a factor of ten difference in intensity. This means that each whole number step represents a ten-fold increase in intensity. Therefore a magnitude 9 earthquake is ten time larger than a magnitude 8 earthquake, one hundred times larger than a magnitude 7 earthquake, and one thousand times larger than a magnitude 6 earthquake.

The magnitude of an earthquake is given by the formula $M = \log\left(\frac{I}{I_o}\right)$, where I is the earthquake intensity and I_o is a reference intensity.

Changing to exponential form provides the formula $\frac{I_1}{I_2} = 10^{M_1 - M_2}$

Ex. How many times more intense was the 1985 Mexico earthquake (8.1) then the 1966 Turkey quake (6.9)?

$10^{8.1-6.9} = 10^{1.2} = 15.8$ times

pH Scale:

The pH scale measures the range of hydrogen ion concentration by determining the acidity or the alkalinity of a solution. The scale measures from 0 to 14 with values below 7 representing increasing acidity, and values above 7 representing increasing alkalinity. The value of 7 represents the neutral level on the pH scale where the solution is neither acidic nor alkaline.

Similar to the Richter scale, the pH scale is logarithmic – a difference in one unit of pH corresponds to a factor of ten difference in intensity. This means that each whole number step represents a ten-fold increase in intensity.

Formula for pH:

The pH of a solution is defined as $pH = -\log(H^+)$, where the H^+ is the hydrogen ion concentration (expressed as mole/litre)

Ex. A patient gave a urine sample which was found to have a pH of 5.7. What was the hydrogen ion concentration?

$5.7 = -\log H \Rightarrow -5.7 = \log H$

$10^{-5.7} = H$

1.995×10^{-6}

Polynomial Functions

Learning Outcomes:

- *To describe, orally and in written form, the characteristics of a polynomial function by analyzing its graph*
- *To describe, orally and in written form, the characteristics of a polynomial function by analyzing its equation.*
- *To match equations in a given set to their corresponding graphs.*
- *To graph data, and determine the polynomial function that best approximates the data*
- *To interpret the graph of a polynomial function that models a situation, and explain the reasoning*
- *To solve, using technology, a contextual problem that involves data that is best represented by graphs of polynomial functions, and explain the reasoning.*

Polynomial Function: A function that contains only the operations of multiplication and addition with real-number coefficients, whole-number exponents, and two variables.
For example: $y=5x^3-3x+7$, $y=-4x+8$, $f(x)=x^2-3x+4$

Degree: The highest exponent in a polynomial function.
For example: The degree of $y=4x^3-8x^2+5$ is '3'.

Leading Coefficient: The coefficient of the term with the greatest degree in the polynomial function in standard form.
For example: The leading coefficient in the function $f(x)=-5x^2+8x-7$ is '-5'.

Constant Term: The term in the polynomial function that has no variable, i.e. the degree is '0'.
For example: The constant term in the function $y=4x^3-6x^2-7x+1$ is '1'.

Linear Function	Quadratic Function	Cubic Function
A polynomial function of first degree, whose greatest exponent is '1'.	A polynomial function of second degree, whose greatest exponent is '2'.	A polynomial function of third degree, whose greatest exponent is '3'.
Standard Form is $f(x)=ax+b$, where $a\neq 0$.	Standard Form is $f(x)=ax^2+bx+c$, where $a\neq 0$.	Standard Form is $f(x)=ax^3+bx^2+cx+d$, where $a\neq 0$.
Examples: $y=2x-3$, $f(x)=-\frac{2}{3}x+4$	Examples: $y=x^2-3x+4$, $f(x)=-\frac{1}{2}x^2+4x+5$	Examples: $y=x^3-3x+2$, $f(x)=-2x^3+4x^2+\frac{3}{4}$

End Behaviour: The behaviour of the y-values of the function as $|x|$ becomes very large.

Turning Point: Any point where the graph of a function changes from increasing to decreasing or decreasing to increasing.

Polynomial Function: a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots a_2x^2 + a_1x + a_0$.

Ex. Identify whether each function is a polynomial function. Justify your answer. State the degree, the leading coefficient, and the constant term of each polynomial function.

a. $h(x) = \frac{1}{x}$ NO

b. $y = 3x^2 - 2x^5 + 4$ yes

c. $y = -4x^4 - 4x + 3$ yes

d. $y = x^{\frac{3}{2}} - 7$ no

Ex. a. Describe the end behavior of the graph of the function $f(x) = -x^3 - 3x^2 + 2x + 1$. State the possible number of x-intercepts, the y-intercept, and whether the graph has a maximum or minimum value (do not graph on calculator).

cubic ⇒ either 3 to 1 or 2 to 4

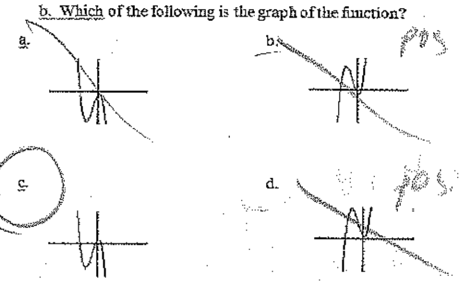
$\boxed{\text{neg} \Rightarrow 2 \text{ to } 4}$

x-int # 3

y-int = 1

no max / min.

b. Which of the following is the graph of the function?

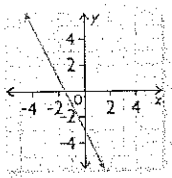


	#of x-intercepts	y-intercept	End Behavior	Domain Range	# of possible turning points
a. $f(x) = 3x - 5$	1	-5	3 to 1	D $x \in \mathbb{R}$ R $y \in \mathbb{R}$	0
b. $f(x) = -2x^2 - 4x + 8$	2	8	3 to 4	D $x \in \mathbb{R}$ R $y \le$ max y	1

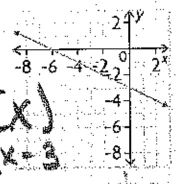
Match each graph with the correct polynomial function.
Justify your reasoning.

$g(x) = -x^3 + 4x^2 - 2x - 2$ $j(x) = x^2 - 2x - 2$ $p(x) = x^3 - 2x^2 - x - 2$

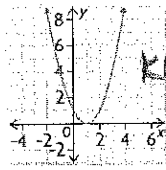
$h(x) = -\frac{1}{2}x - 3$ $k(x) = x^2 - 2x + 1$ $q(x) = -2x - 3$

i)


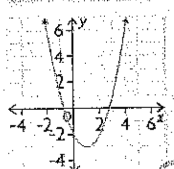
q(x)
$-2x-3$

iii)


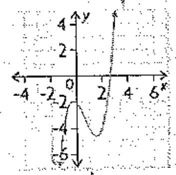
h(x)
$-\frac{1}{2}x-3$

v)


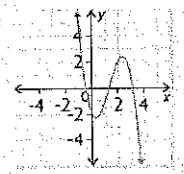
$k(x)=x^2-2x+1$

ii)


$j(x)=x^2-2x-2$

iv)


p(x)
pos.
cubic

vi)


neg cubic
g(x)

Sinusoidal Functions

Learning Outcomes:

- *To estimate and determine benchmarks for angle measure*
- *To describe, orally and in written form, the characteristics of a sinusoidal function by analyzing its graph*
- *To investigate the characteristics of the graphs of sine and cosine functions*
- *To describe, orally and in written form, the characteristics of a sinusoidal function by analyzing its graph*
- *To identify characteristics of the graphs of sinusoidal functions*
- *To describe, orally and in written form, the characteristics of a sinusoidal function by analyzing its equation*
- *To match equations in a given set to their corresponding graphs*
- *To graph data, and determine the sinusoidal function that best approximates the data*
- *To interpret the graph of a sinusoidal function that models a situation, and explain the reasoning*
- *To solve, using technology, a contextual problem that involves data that is best represented by graphs of sinusoidal functions, and explain the reasoning*

Radian measure is an alternate way to express the size of an angle. Using radians allows you to express the measure of an angle as a real number without units.

Conversion Chart

Degrees to Radians multiply by $\frac{\pi}{180}$ Radians to Degrees multiply by $\frac{180}{\pi}$

Ex. Change each degree to radian and each radian to degree:

a. 30°

$30 \times \frac{\pi}{180} = \frac{\pi}{6}$

b. $\frac{2\pi}{3}$

$\frac{2\pi}{3} \times \frac{180}{\pi} = 120°$

c. 210°

$210 \times \frac{\pi}{180} = \frac{7\pi}{6}$

d. $\frac{9\pi}{5}$

$\frac{9\pi}{5} \times \frac{180}{\pi} = 324°$

The central angle formed by one complete revolution in a circle is 360°, or 2π in radian measure.

Need to know:

- Use benchmarks to estimate the degree measure of an angle given in radians
- In radian measure,
  - 1 is equivalent to about 60°
  - π is equivalent to 180°
  - 2π is equivalent to 360°

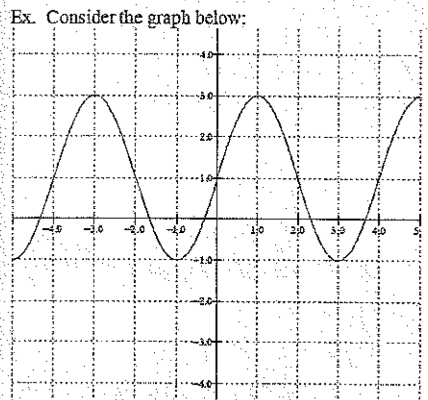
Periodic Functions: A periodic function is a function whose graph repeats regularly over some interval of the domain.

Period: The length of the interval of the domain to complete one cycle

Midline: the horizontal line halfway between the maximum and minimum values of a periodic function. $y = \frac{\textit{maximum value} + \textit{minimum value}}{2}$

Amplitude: the distance from the midline to either the maximum or minimum value of a periodic function; the amplitude is always expressed as a positive number. $\textit{Amplitude} = \frac{\textit{maximum value} - \textit{minimum value}}{2}$

Ex. Consider the graph below:



Determine the midline, amplitude, period, domain and range of the graph.

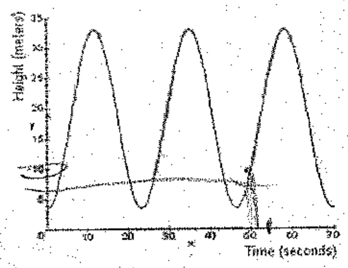
midline $= \frac{3 + -1}{2} = \boxed{1}$

amp $= \frac{3 - -1}{2} = \boxed{2}$

d: $x \in \mathbb{R}$

R: $-1 \leq y \leq 3, y \in \mathbb{R}$

Ex. The graph shows the height, h metres, above the ground over time, t, in seconds it takes a person in a chair on a ferris wheel to complete 3 revolutions. The minimum height of the ferris wheel is 4m and the maximum height is 33m.



Estimate a person's height above the ground at 52 seconds into the ride.

Approx 10m.

Sinusoidal functions: any periodic function whose graph has the same shape as that of $y = \sin x$

Sinusoidal functions can be used as models to solve problems that involve repeating or periodic behavior.

Sinusoidal function is given by:

$$y = a \sin b(x - c) + d$$

Where a, b, c, and d are real numbers.

Summary of the Effect of the Parameters a, b, c and d:

For: $y = a\sin\left[b(x-c)\right]+d$
$y = a\cos\left[b(x-c)\right]+d$

amplitude $= |a| = \frac{\max - \min}{2}$

Period $= \frac{360°}{|b|}$ (for degrees) X

Period $= \frac{2\pi}{|b|}$ (for radians)

Horizontal phase shift = c

- to right if c > 0
- to left if c < 0

Vertical displacement = d

- up if d > 0, down if d < 0
- $d = \frac{\max + \min}{2}$

Range: minimum value: $d - a$
Maximum value: $d + a$

Ex: $y = 2\sin(2x - 90) + 4$
$= 2\sin 2(x - 45) + 4$

shift right 45°

Ex. Determine the amplitude, equation of the midline, range, and period, as well as the relevant horizontal translation of the following equations, then verify your information graphically.

a. $y = 3\cos 3x + 1$ in degrees mode [0, 360, 90][-4, 4, 1]

$a = 3$

midline $y = 1$

Range: $1+3=4$ $\quad {}^-2 \le y \le 4$

$1-3 = {}^-2$

period: $\frac{360}{3} = 120°$

no horiz translation

b. $y = 2\sin 4(x - 45°)$

$a = 2$

period $= \frac{360}{4} = 90°$

mid $y = 0$

range $= {}^-2 \le y \le 2$

shift 45° to right

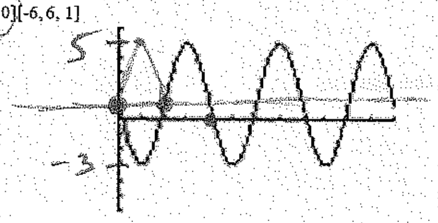
Ex. Given the following equations:

i) $y = 4\cos(x - 90°) + 1$ ii) $y = 5\sin 3(x - 60°)$

iii) $y = 4\sin 3(x - 60°) + 1$ iv) $y = 4\cos 3(x - 60°) + 1$

Match one of the equations with the following graphs:

a. [0, 360, 30][-6, 6, 1]



$a = \frac{5 - {}^-3}{2} = 4$

period $=$ $b = \frac{360}{120} = 3$

shifted 60° to right

intersects mid line at 60° so original would be on y-axis which means sin

Ex. The minimum depth of water in a harbour can be approximated by the function
$h(t) = 12 + 5\cos 0.5t$ where $0 \le t \le 24$

Radian Mode

a. Determine the max and min values:

$12 + 5 = 17$

$12 - 5 = 7$

b. Period of the function:

$\frac{2\pi}{0.5} = 12.6$ hrs.

c. Suitable Window to view graph:

$x:[0,24,2]\ y:[0,20,5]$

d. What is the depth of water, at 2:00 am?

$x = 2 \quad y = 14.7$ m

e. A ship requires 8.5 m of water in the harbour to dock safely. It is midnight, when will the boat have to move to prevent grounding?

$x = 0$

$y_2 = 8.5$

intersect 4.7 in approx 5 hrs.

f. When can the ship return to the harbour?

next intersection

7.87 → about 8am

Regressions

Consider two set of data values, x and y. Plotting the data values on a graph results in a set of points. If the points appear to follow a pattern, (eg. linear, quadratic, exponential, logarithmic, trigonometric, etc.) the technique of regression can be used to fit the best possible curve for the data.

We can use a graphing calculator to:

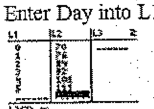
- enter the data in lists
- plot the data
- calculate the equation of the type of curve which best fits the data
- graph the curve to represent the data plot

Using the data from the explore section:

On calculator: STAT:1:Edit (this is where we enter the data into lists)

```
EDIT CALC TESTS
1:Edit...
2:SortA(
3:SortD(
4:ClrList
5:SetUpEditor
```

Enter Day into L1 and Number of Flies into L2



To find the regression equation: STAT:CALC:0:ExpReg

```
EDIT CALC TESTS
7↑QuartReg
8:LinReg(a+bx)
9:LnReg
0:ExpReg
A:PwrReg
B:Logistic
C:SinReg
```

Press enter (be sure to tell the calculator which lists to use)

```
ExpReg L1,L2
```

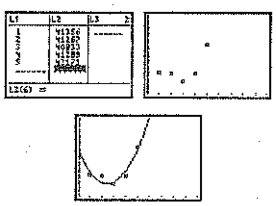
```
ExpReg
 y=a*b^x
 a=69.68515727
 b=1.097283274
```

Curve of best fit: A curve that best approximates the trend on a scatter plot.

Ex. The table below lists the total estimated numbers of AIDS cases, by year of diagnosis from 2007 to 2011 in the United States (Source: US Dept. of Health and Human Services, Centers for Disease Control and Prevention, HIV/AIDS Surveillance)

t	Year	AIDS Cases
1	2007	41,356
2	2008	41,267
3	2009	40,833
4	2010	41,289
5	2011	43,171

a. Plot the data on a scatter plot. Determine the equation of a quadratic regression function for the data.



$y = 345.14x^2 - 1705.66x + 42903.6$

b. Predict the cumulative number of AIDS cases for the year 2014. $t = 8$

$y = 51347$

c. In what year will the number of cumulative cases of AIDS exceed 60, 000?

y_1 = regression equation
y_2 60000 calc intersect. $x = 9.9$ or 10 years

Ex. The following data are water levels recorded on November 18 and 19, 2005 at Oak Bay Station in British Columbia. Time is in hours and depth is in meters

Time	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28
Depth	1.0	0.6	0.9	1.6	2.4	2.9	2.9	2.6	2.5	2.4	2.3	2.0	1.5	0.9	1.0

a) What is the sinusoidal regression equation for the above data? Set an appropriate window and sketch the graph produced below.

$y = 1.1\sin(0.21x - 1.39) + 1.84$

b) Use the graph or the equation to estimate the maximum, minimum, and median values, the amplitude, and the period.

max: 2.94 amp: 1.1
min: 0.74 period: 29.5
median: 1.84 ?midline

b) Use your **graph** to estimate the depth of the water at 6:30 pm on November 18. Is the tide coming in or going out at this time?

$x = 18.5$ → 18:30 hours

$y = 2.5$ tide is going out

c) Between what times is the depth of the water more than 2.5 m?

9.5 to 18.3 9:30am and 6:13pm

Rational Expressions & Equations

Learning Outcomes:

- *To compare the strategies for writing equivalent forms of rational expressions to writing equivalent forms of rational numbers*
- *To be able to explain why a given value is non-permissible for a given rational expression*
- *To determine the non-permissible values for a rational expression*
- *To determine a rational expression that is equivalent to a given rational expression by multiplying the numerator and denominator by the same factor, and state the non-permissible values of the equivalent rational expression*
- *To be able to simplify a rational expression*
- *To explain why the non-permissible values of a given rational expression and its simplified form are the same*
- *To identify and correct errors in a given simplification of a rational expression, and explain the reasoning*
- *To develop strategies for multiplying rational expressions*
- *To determine the non-permissible values when performing multiplication on rational expressions*
- *To develop strategies for dividing rational expressions*
- *To determine the non-permissible values when performing division on rational expressions*
- *To determine, in simplified form, the sum or difference of rational expressions that have the same denominator*
- *To determine, in simplified form, the sum or difference of rational expressions that have different denominators*
- *To determine the non-permissible values when performing adding and subtracting on rational expressions*
- *Identify non-permissible values in a rational equation*
- *Determine the solution to a rational equation algebraically*
- *Solve problems using a rational equation*

Rational Expression: an algebraic fraction with a numerator and a denominator that are polynomials.

Can you provide an example of a rational expression?

Whenever you use a rational expression, you must identify any values that must be excluded or are considered non-permissible values. Non-permissible values are all values that make the denominator zero.

Ex. Determine the non-permissible value(s) for each rational expression:

a. $\frac{4a}{3bc}$ $b, c \neq 0$

b. $\frac{x-1}{(x+2)(x-3)}$ $x \neq 2, -3$

Equivalent Expressions

Ex. a. Write a rational number that is equivalent to $\frac{8}{12}$

$\frac{4}{6}, \frac{2}{3}, \frac{16}{24}$

b. Write a rational expression that is equivalent to $\frac{4x^2+8x}{4x}$

$\frac{2x+4}{2}$ or $\frac{8x^2+16x}{8x}$

In order to make an equivalent expression, what process do we need to follow?

$\times$ or $\div$

Ex. For each of the following, determine if the rational expressions are equivalent.

a. $\frac{9^{(-2)}}{3x-1^{(-2)}}$ *and* $\frac{-18}{2-6x}$

$\frac{-18}{-6x+2}$ ✓

b. $\frac{2-2x}{4x}$ *and* $\frac{x-1}{2x}$

$\div 2$

$\frac{1-x}{2x}$ NO

Simplifying Rational Expressions

A rational expression is in simplified form when the numerator and denominator have no common factors

Always state the non-permissible values of the variables as restrictions before simplifying a rational expression. Otherwise, if you eliminate a variable or factor as you simplify, you will lose the information about the non-permissible value of the variable.

$\frac{15x^3y^4}{-5xy^3}$ $x, y \neq 0$

$\boxed{3x^2y}$

$\frac{5x^2-10x}{5x}$ $x \neq 0$

$\boxed{x-2}$

$\frac{3a^3-3a^2}{-12a+12}$ $a \neq 1$

$\frac{3a^2(a-3)}{-12(a-12)}$

$\boxed{\frac{a^2(a-3)}{-4(a-12)}}$

MULTIPLYING RATIONAL EXPRESSIONS

Basic Steps:

1. Factor all expressions
2. State restrictions on the denominator
3. Cancel out any common factors

Ex. Multiply and simplify the following fractions:

a. $\left(\frac{\cancel{5}^{1}}{\cancel{8}_{2}}\right)\left(\frac{\cancel{4}^{1}}{\cancel{15}_{3}}\right) = \frac{1}{6}$

b. $\left(\frac{\cancel{4x}}{3\cancel{xy}}\right)\left(\frac{y^4}{\cancel{8x}}\right)$ $\frac{y}{6}$ $x, y \neq 0$

Ex. Simplify the following products:

a. $\frac{2x^2-12x}{15x} \cdot \frac{5x}{x-6}$ $\frac{2x\cancel{(x-6)}}{3\cancel{15x}} \cdot \frac{\cancel{5x}}{\cancel{(x-6)}} = \frac{2x}{3}$ $x \neq 0, 6$

DIVIDING RATIONAL EXPRESSIONS

Basic Steps:

1. Factor all expressions
2. Take the reciprocal of the second expression and multiply
3. State all restrictions, for the second expression, be sure to look at the numerator and the denominator for all restrictions. (both will actually spend time as the denominator and therefore need to be stated as restrictions)
4. Cancel out any common factors

Ex. Simplify each quotient

a. $\frac{2w}{24w+4w^2} \div \frac{6w^2-6w}{9w^3+54w^2}$ $\boxed{w \neq 0, -6, 1}$

$\frac{\cancel{2w}}{\cancel{4w}(6+w)} \cdot \frac{9w^2(w+6)}{\cancel{6w}(w-1)}$

$\boxed{\frac{3w}{4}}$

$\frac{4x^2-1}{x+2} \div \frac{4x^2+2x}{8x^2-32}$ $\boxed{x \neq \pm 2, 0, -\frac{1}{2}}$

$2x(2x+1)$, $8(x^2-4)$

$\frac{(2x-1)\cancel{(2x+1)}}{\cancel{(x+2)}} \times \frac{8\cancel{(x+2)}(x-2)}{2x\cancel{(2x+1)}}$

$\boxed{\frac{4(2x-1)(x-2)}{x}}$

ADDING AND SUBTRACTING RATIONAL EXPRESSIONS

Steps:

1. Factor all expressions
2. State all restrictions on the denominator
3. Cancel any common factors
4. Find a common denominator
5. Change to equivalent fractions with the common denominator
6. Add or subtract the numerators
7. Reduce if possible

Ex. Determine each sum or difference. Express each answer in simplest form. Identify all non-permissible values.

a. $\frac{2x-1}{6} + \frac{3x+2}{4} = \frac{4(2x-1)+6(3x+2)}{24}$

$\frac{8x-4+18x+12}{24}$

$\frac{26x+8}{24} = \frac{2(13x+4)}{24_{12}} = \boxed{\frac{13x+4}{12}}$

b. $\frac{3}{3x+9} - \frac{16x}{4x-12}$

$3(x+3)$ $4(x-3)$

$\frac{3(4)(x-3) - 4\cdot 16x(3)(x+3)}{12(x+3)(x-3)} = \frac{(x-3)-4(x+3)}{(x+3)(x-3)} = \frac{-3x-15}{(x+3)(x-3)} = \boxed{\frac{-3(x+5)}{(x+3)(x-3)}}$ $\boxed{x \neq \pm 3}$

c. $\frac{2x}{x^2-1} - \frac{4}{x-1}$

$\frac{2x-4(x+1)}{(x+1)(x-1)} = \frac{2x-4x-4}{(x+1)(x-1)}$

$-2x-4 = \boxed{\frac{-2(x+2)}{(x+1)(x-1)} \quad x \neq \pm 1}$

SOLVING RATIONAL EQUATIONS

Rational equations can be used to solve several different kinds of problems, such as work-related problems, where two people or machines work together at different rates to complete a task.

Working with a rational equation is similar to working with rational expressions. A significant difference occurs because in an equation, what you do to one side you must also do to the other side.

To solve a rational equation:

1. Factor each denominator
2. Identify the non-permissible values
3. Multiply both sides of the equation by the lowest common denominator
4. Solve by isolating the variable on one side of the equation
5. Check your answers

$$\frac{3}{x+2}-\frac{5}{x-2}=\frac{2}{x^2-4}$$ *Need to factor first:*

$$\frac{3}{x+2}-\frac{5}{x-2}=\frac{2}{(x+2)(x-2)}$$ *LCD is (x+2)(x-2)!*

Multiply by the LCD by the values that are uncommon to each denominator:

$(x-2)(3)-(x+2)(5)=2$

$3x-6-5x-10=2$ Check: both sides equal to 0.025

$-2x=18$

$x=-9$

Ex. Solve the equation. What are the non-permissible values?

$$(x+2)(x-3)\left[\frac{3x}{x+2}-\frac{5}{x-3}=\frac{-25}{\underset{(x+2)(x-3)}{x^2-x-6}}\right]$$

$\boxed{x \neq -2, 3}$

$3x(x-3)-5(x+2)=-25$

$3x^2-9x-5x-10=-25$

$3x^2-14x+15=0$

$\frac{-9}{}\ \frac{-5}{} = 45$

$-9+-5=-14$

$\frac{(3x-9)(3x-5)}{3}=0$

$(x-3)(3x-5)$

$x \neq 3 \quad \boxed{\frac{5}{3}}$

reject as extraneous root

$$\text{Ex.}\left[\frac{18}{x^2-3x}=\frac{6}{x-3}-\frac{5}{x}\right]x(x-3)$$

$18=6x-5(x-3)$

$18=6x-5x+15$

$18=x+15$

$\boxed{3=x}$ but $x \neq 3$ so no solution

APPLICATIONS OF RATIONAL EQUATIONS

Ex. Rima bought a case of concert t-shirts for $450. She kept two t-shirts for herself and sold the rest for $560, making a profit of $10 on each shirt. How many t-shirts were in the case?

Buying price per t − shirt: $\frac{450}{x}$

Selling price per T − shirt: $\frac{560}{x-2}$

Profit = Selling price − Buying price

bought x

sold $x-2$

?

$$10=\frac{560}{x-2}-\frac{45}{x}$$

$10(x-2)(x)=560x-45(x-2)$

$10x^2-20x=560x-45x+90$

$10x^2-535x-90=0$

$\boxed{x=53.7}$

A plane flew from Red Deer to Winnipeg, a flying distance of 1 260 km. On the return journey, due to a strong head wind, the plane travelled 1 200 km in the same time it took to complete the outward journey. On the outward journey, the plane was able to maintain an average speed 20 km/hr greater than on the return journey.

- If the average speed of the plane from Red Deer to Winnipeg is x km/hr, state an expression for the average speed of the plane from Winnipeg to Red Deer in km/hr.

$x-20$

- Write an expression in x for the time taken to fly the 1 260 km from Red Deer to Winnipeg.

$t=\frac{d}{s}=\frac{1260}{x}$

- Write an expression in x for the time taken to fly 1 200 km on the return journey.

$\frac{1200}{x-20}$

- Calculate the average speed of the plane from Winnipeg to Red Deer.

time is equal so $\frac{1200}{x-20}=\frac{1260}{x}$

$1200x = 1260x - 25200$

$25200 = 60x$

$\boxed{420 = x}$ (Red Deer to Winnipeg)

Winnipeg to Red Deer: $x-20 = 420-20 = \boxed{400 \text{ Km/h}}$

	d	s	t
Win to RD	1200	$x-20$	$\frac{1200}{x-20}$
RD to Win	1260	x	$\frac{1260}{x}$