Math 10 C: Exponents & Radicals

Concept 1: Prime Factorization

Vocabulary

Factors: Factors are the numbers (or terms) that multiply together to get another number.

Prime Number: a whole number with exactly two distinct factors; itself and 1.

Example: 3 is a prime number because it only has two factors – itself and 1

Composite Number: a whole number with three or more factors; a number that is not a prime number.

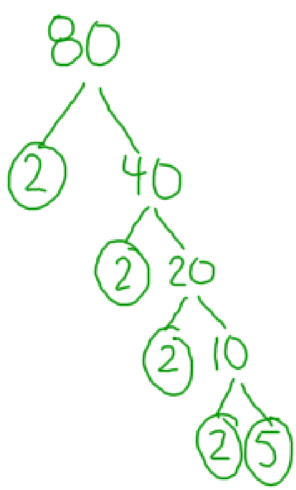
Example: 4 can be written as 1×4 and 2×2. Therefore 4 has three factors: 1, 2 and 4 and is a composite number.

Prime Factors: A factor that is a prime number. One of the prime numbers that, when multiplied, give the original number.

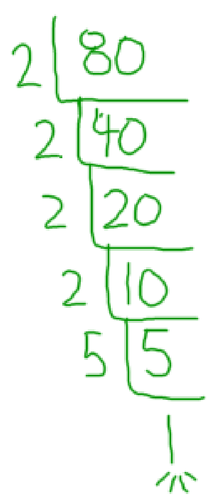
Prime Factorization: writing a number as a product of its prime factors.

Skills:

Factor Tree:



Birthday Cake:



Therefore,

$$80 = 2 \bullet 2 \bullet 2 \bullet 2 \bullet 5$$
$$80 = 2^4 \bullet 5$$

Concept 2: Perfect Squares and Square Roots

Vocabulary & Key Concepts

Perfect Square – a number that can be expressed as the product of two equal integral factors

For example) 9 is a perfect square because $3\times3=9$ or $-3\times-3=9$

Square Root – one of the two equal factors of a number

For example) $\sqrt{49}=\sqrt{7\times7}$, therefore 7 is the square root of 49.

Notation:

$$\sqrt{x}=x^{\frac{1}{2}}$$

For example)

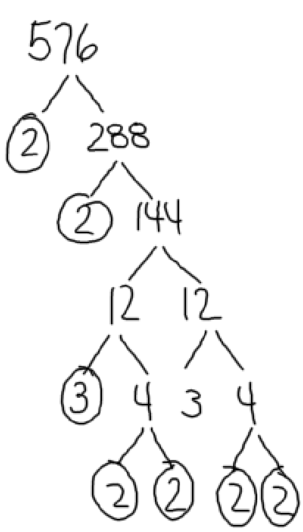
$$\begin{aligned}&\sqrt{49}\\ &=49^{\frac{1}{2}}\\ &=7\end{aligned}$$

Skills

Using Prime Factorization to determine if #'s are perfect squares.

Example:

Is 576 a perfect square?



• Checking if 576 is a perfect square so need two equal groupings of prime factors.

$576=(2\cdot2\cdot2\cdot3)(2\cdot2\cdot2\cdot3)$

$\therefore \sqrt{576}=2\cdot2\cdot2\cdot3$

$=24$

Evaluate involving the squaring of numbers.

To evaluate squaring a number the exponent 'goes to' what it is directly beside.

For examples:

$$\begin{aligned}&(-3)^2 \\ &= (-3)(-3) \\ &= 9\end{aligned} \qquad \text{vs.} \qquad \begin{aligned}&-3^2 \\ &= -3\times 3 \\ &= -9\end{aligned}$$

Evaluating products under square root signs:

$$\begin{aligned}&\sqrt{(16)(9)} \\ &= \sqrt{16}\times\sqrt{9} \\ &= 4\times 3 \\ &= 12\end{aligned}$$

Note: Easier to mentally find the square roots of smaller numbers then multiplying the values together and then taking the square root of the larger number.

Estimating Square Roots

To estimate: use the values of perfect squares that you know as your 'benchmarks' for approximating the values in between.

Memorize the perfect squares (up to 100)

$\sqrt{4} = 2; \sqrt{9} = 3; \sqrt{16} = 4; \sqrt{25} = 5; \sqrt{36} = 6; \sqrt{49} = 7; \sqrt{64} = 8; \sqrt{81} = 9; \sqrt{100} = 10$

Example: Approximate the value of $\sqrt{30}$ to the nearest tenth.

$\sqrt{30}$ is between $\sqrt{25}$ and $\sqrt{36}$ so the value must be between 5 and 6.

The approximate value of $\sqrt{30}$ is 5.4.

Check your Understanding:

1. Use prime factorization to determine if 196 is a perfect square.
2. Evaluate $256^{\frac{1}{2}}$ using prime factorization.
3. Evaluate $\sqrt{(25)(4)}$
4. Approximate the value of $\sqrt{74}$ to the nearest tenth.

Concept 3: Perfect Cubes and Cube Roots

Vocabulary & Key Concepts

Perfect Cube – a number that can be expressed as the product of three equal integral factors

For example) 8 is a perfect cube because $2 \times 2 \times 2 = 8$

Cube Root – one of the three equal factors of a number

For example) $\sqrt[3]{27} = \sqrt[3]{3 \times 3 \times 3}$, therefore 3 is the cube root of 27.

Notation: $\sqrt[3]{x} = x^{\frac{1}{3}}$

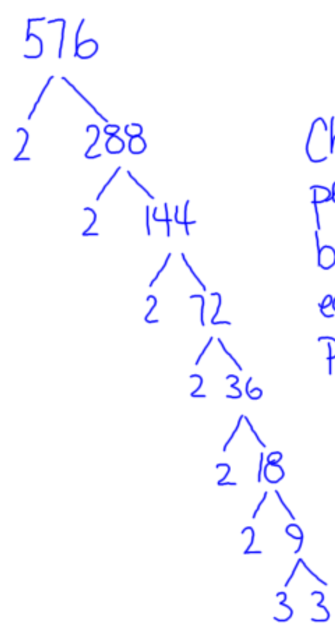
For example)
$$\begin{aligned} &\sqrt[3]{27} \\ &= 27^{\frac{1}{3}} \\ &= 3 \end{aligned}$$

Skills

Using Prime Factorization to determine if #'s are perfect cubes.

Example:

Is 576 a perfect cube?



Checking if 576 is a perfect cube, need to be able to make 3 equal groups with prime factors

$576 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$

Cannot make 3 equal groups so <u>not</u> a perfect cube (it <u>is</u> a perfect square)

Evaluate involving the cubing of numbers.

To evaluate squaring a number the exponent 'goes to' what it is directly beside.

For examples:
$$\begin{aligned}&(-3)^3\\&=(-3)(-3)(-3)\\&=-27\end{aligned}\qquad \text{vs.}\qquad \begin{aligned}&-3^3\\&=-3\times3\times3\\&=-27\end{aligned}$$

Note: Answers are the same regardless of whether there are brackets or not.

Evaluating products under cube root signs:

$$\begin{aligned}&\sqrt[3]{(64)(27)}\\&=\sqrt[3]{64}\times\sqrt[3]{27}\\&=4\times3\\&=12\end{aligned}$$

Note: Easier to mentally find the cube roots of smaller numbers then multiplying the values together and then taking the cube root of the larger number.

Estimating Cube Roots

To estimate: use the values of perfect cubes that you know as your 'benchmarks' for approximating the values in between.

Memorize the following perfect cubes :

$\sqrt[3]{1}=1; \sqrt[3]{8}=2; \sqrt[3]{27}=3; \sqrt[3]{64}=4; \sqrt[3]{125}=5; \sqrt[3]{216}=6; \sqrt[3]{343}=7$

Example: Approximate the value of $\sqrt[3]{95}$ to the nearest tenth.

$\sqrt[3]{95}$ is between $\sqrt[3]{64}$ and $\sqrt[3]{125}$ so the value must be between 4 and 5.

The approximate value of $\sqrt[3]{95}$ is 4.5.

Check your Understanding:

1. Determine if 4096 is a perfect cube (use prime factorization).
2. Evaluate $\sqrt[3]{\dfrac{27}{125}}$
3. Approximate the value of $\sqrt[3]{74}$ to the nearest tenth.

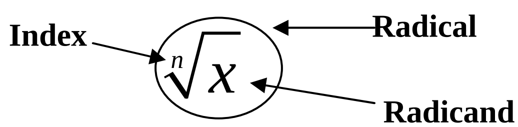
Concept 4: Mixed and Entire Radicals

Vocabulary & Key Concepts

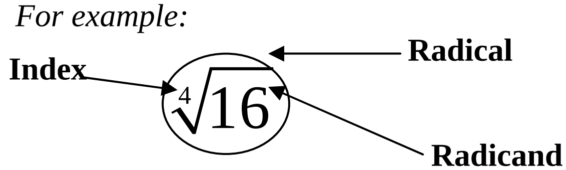
Radicals (sometimes called the ‘root’): an expression that consists of a root symbol, an index and a radicand

Parts of a radical:

In general:



For example:



Radicand – the quantity or expression under the radical sign

Index – indicates what root to take

Mixed Radical – the product of a rational number and a radical. Eg. $3\sqrt{5}$

Entire Radical – the product of 1 and a radical. Eg. $\sqrt{50}$

Skills

Converting from Entire Radical to a Mixed Radical

Entire radicals can be written as mixed radicals if the radicand has a factor that is a perfect square or perfect cube.

Note: Not all entire radicals can be written as mixed radicals.

Recall: Perfect Squares → 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, etc.
Perfect Cubes → 8, 27, 64, 125, 216, etc.

Ex) Convert $\sqrt{24}$ into a mixed radical.

$\sqrt{24}$ → Factors of 24: 1, 24
2, 12
3, 8
4, 6 → 4 is a perfect square

$= \sqrt{24}$
$= \sqrt{4 \times 6}$
$= \sqrt{4} \times \sqrt{6}$
$= 2\sqrt{6}$

Ex) Convert $\sqrt[3]{54}$ into a mixed radical.

$\sqrt[3]{54}$ → Factors of 54: 1, 54
2, 27 → 27 is a perfect cube

$$\begin{aligned}&\sqrt[3]{54}\\&=\sqrt[3]{27\times 2}\\&=\sqrt[3]{27}\times\sqrt[3]{2}\\&=3\times\sqrt[3]{2}\\&=3\sqrt[3]{2}\end{aligned}$$

Converting from Mixed Radical to an Entire Radical

Both numbers need to be written as radicals so that we can multiply and simplify.

Note: ALL mixed radicals can be written as entire radicals.

Ex) Write $4\sqrt{3}$ as an entire radical.

$$\begin{aligned}&4\sqrt{3}\\&=\sqrt{4^2}\times\sqrt{3}\\&=\sqrt{16}\times\sqrt{3}\\&=\sqrt{48}\end{aligned}$$

Ex) Write $3\sqrt[3]{4}$ as an entire radical.

$$\begin{aligned}&3\sqrt[3]{4}\\&=\sqrt[3]{3^3}\times\sqrt[3]{4}\\&=\sqrt[3]{27}\times\sqrt[3]{4}\\&=\sqrt[3]{27\times 4}\\&=\sqrt[3]{108}\end{aligned}$$

<u>Check your Understanding:</u>

1. Express the following as mixed radicals.

(a) $\sqrt{45}$ (b) $\sqrt[3]{16}$ (c) $80^{\frac{1}{2}}$

2. Express the following as entire radicals.

(a) $4\sqrt[3]{2}$ (b) $2\sqrt{8}$ (c) $2\sqrt[5]{3}$

Concept 5: Exponent Laws (with Rational Bases)

Vocabulary & Key Concepts

Recall: Laws of Exponents (from Grade 9)

- Multiplying terms with the same base – add the exponents
  In general form: $x^m \times x^n = x^{m+n}$ Example: $2^2 \times 2^5$
  $= 2^{2+5}$
  $= 2^7$

- Dividing terms with the same base – subtract the exponents
  In general form: $x^m \div x^n = x^{m-n}$ Example: $3^5 \div 3^2$
  $= 3^{5-2}$
  $= 3^3$

- Power to a power – multiply the exponents
  In general form: $(x^m)^n = x^{m \times n}$ Example: $\left(4^2\right)^4$
  $= 4^{2\times 4}$
  $= 4^8$

- Power of Zero
  In general form: $a^0 = 1$, where $a \neq 0$
  Examples: (a) $(-6)^0 = 1$ (b) $-6^0 = -1$ (c) $\dfrac{1}{-5^0 - (-5)^0} = \dfrac{1}{-1-1}$
  $= \dfrac{1}{-2}$

Skills

Evaluating Powers of the Form $a^{\frac{1}{n}}$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Examples:

Write the following expressions as radicals, and then evaluate.

(a) $64^{\frac{1}{3}}$
$= \sqrt[3]{64}$
$= 4$

(b) $(-27)^{\frac{1}{3}}$
$= \sqrt[3]{-27}$
$= -3$

(c) $(16)^{\frac{1}{4}}$
$= \sqrt[4]{16}$
$= 2$

(d) $\left(\dfrac{9}{16}\right)^{\frac{1}{2}}$
$= \left(\dfrac{\sqrt[2]{9}}{\sqrt[2]{16}}\right)$
$= \dfrac{3}{4}$

Evaluating Powers with Rational Exponents and Rational Bases

$$a^{\frac{m}{n}} = \sqrt[n]{a}^{\,m} \quad OR \quad a^{\frac{m}{n}} = \sqrt[n]{(a)^m}$$

Examples:

Rewrite the following in another form, then evaluate.

(a) $8^{\frac{2}{3}}$

$$= (8^2)^{\frac{1}{3}}$$
$$= \sqrt[3]{8^2}$$
$$= \sqrt[3]{64}$$
$$= 4$$

or

$$8^{\frac{2}{3}}$$
$$= (8^{\frac{1}{3}})^2$$
$$= \left(\sqrt[3]{8}\right)^2$$
$$= \left(2\right)^2$$
$$= 4$$

(b) $25^{\frac{3}{2}}$

$$= \sqrt[2]{25^3}$$
$$= \left(\sqrt[2]{25}\right)^3$$
$$= 5^3$$
$$= 125$$

(c) $\left(\frac{1}{4}\right)^{\frac{5}{2}}$

$$= \left(\frac{\sqrt{1}}{\sqrt{4}}\right)^5$$
$$= \left(\frac{1}{2}\right)^5$$
$$= \frac{1}{32}$$

Evaluating Powers with Negative Integer Exponents

$$a^{-n} = \frac{1}{a^n}\text{, where } a \neq 0$$

Examples:

Evaluate the following:

(a) 3^{-2}

$$= \left(\frac{1}{3}\right)^2$$
$$= \frac{1^2}{3^2}$$
$$= \frac{1}{3}$$

(b) -4^{-2}

$$= -1(4)^{-2}$$
$$= -1\left(\frac{1}{4}\right)^2$$
$$= -1\left(\frac{1}{16}\right)$$
$$= \frac{-1}{16}$$

(c) $\left(\frac{3}{2}\right)^{-2}$

$$= \left(\frac{2}{3}\right)^2$$
$$= \frac{2^2}{3^2}$$
$$= \frac{4}{9}$$

(d) $3^{-2} + 3^{-1}$

$$= \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^1$$
$$= \frac{1}{9} + \frac{1}{3}$$
$$= \frac{1}{9} + \frac{3}{9}$$
$$= \frac{4}{9}$$

Evaluating Powers with Negative Rational Exponents

$$a^{-\frac{m}{n}} = \left(\frac{1}{a}\right)^{\frac{m}{n}}$$
$$= \sqrt[n]{\left(\frac{1}{a}\right)^m}$$

Examples:

Rewrite the following in another form, then evaluate.

(a) $27^{-\frac{2}{3}}$

$=\left(\frac{1}{27}\right)^{\frac{2}{3}}$

$=\left(\frac{\sqrt[3]{1}}{\sqrt[3]{27}}\right)^{2}$

$=\left(\frac{1}{3}\right)^{2}$

$=\frac{1}{9}$

(b) $\left(\frac{1}{16}\right)^{-\frac{3}{2}}$

$=\left(\frac{16}{1}\right)^{\frac{3}{2}}$

$=16^{\frac{3}{2}}$

$=\left(\sqrt{16}\right)^{3}$

$=4^{3}$

$=64$

Check your Understanding:

1. Evaluate the following expression.

   (a) $\left(\frac{9}{16}\right)^{\frac{1}{2}}$ (b) $49^{\frac{3}{2}}$ (c) $\left(\frac{8}{27}\right)^{\frac{2}{3}}$

   (d) $16^{\frac{-1}{2}}$ (e) $\left(\frac{16}{81}\right)^{\frac{-3}{4}}$

2. Write each radical expression with exponents and integral bases.

   (a) $\sqrt[5]{4}$ (b) $\left(\sqrt{5}\right)^{3}$ (c) $\frac{1}{\sqrt[3]{3}}$ (d) $\sqrt[6]{2^{5}}$

Concept 6: Working with Exponents (with Variable Bases)

Vocabulary & Key Concepts

Skills

Simplifying Algebraic Expressions with Integer Exponents

Examples:
Simplify the following expressions. Leave all exponents as positive values.

(a) $x^3y^{-2} \bullet x^{-4}y^3$

$$= x^3x^{-4} \bullet y^{-2}y^3$$
$$= x^{3+-4} \bullet y^{-2+3}$$
$$= x^{-1} \bullet y^1$$
$$= \frac{1}{x^1} \bullet \frac{y^1}{1}$$
$$= \frac{y^1}{x^1} \quad or \quad \frac{y}{x}$$

(b) $\dfrac{4m^3n^{-2}}{5m}$

$$= \frac{4}{5} \bullet \frac{m^3}{m} \bullet n^{-2}$$
$$= \frac{4}{5} \bullet m^{3-1} \bullet \frac{1}{n^2}$$
$$= \frac{4}{5} \bullet m^2 \bullet \frac{1}{n^2}$$
$$= \frac{4m^2}{5n^2}$$

Simplifying Algebraic Expressions with Rational Exponents

Examples:
Simplify the following expressions. Leave all exponents positive

(a) $(25a^4b^2)^{\frac{3}{2}}$

$$= 25^{\frac{3}{2}}\left(a^4\right)^{\frac{3}{2}}\left(b^2\right)^{\frac{3}{2}}$$
$$= \left(\sqrt{25}\right)^3\left(a^{\frac{12}{2}}\right)\left(b^{\frac{6}{2}}\right)$$
$$= 5^3a^6b^3$$
$$= 125a^6b^3$$

(b) $\dfrac{12x^{-5}y^{\frac{5}{2}}}{3x^{\frac{1}{2}}y^{-\frac{1}{2}}}$

$$= \frac{12}{3} \bullet \frac{x^{-5}}{x^{\frac{1}{2}}} \bullet \frac{y^{\frac{5}{2}}}{y^{\frac{-1}{2}}}$$
$$= 4 \bullet x^{-5-\frac{1}{2}} \bullet y^{\frac{5}{2}-\left(-\frac{1}{2}\right)}$$
$$= 4 \bullet x^{\frac{-10}{2}-\frac{1}{2}} \bullet y^{\frac{5}{2}+\frac{1}{2}}$$
$$= 4 \bullet x^{\frac{-11}{2}} \bullet y^{\frac{6}{2}}$$
$$= 4x^{\frac{-11}{2}}y^3$$
$$= \frac{4y^3}{x^{\frac{11}{2}}}$$

Changing from Radical to Exponential Form

Examples:

Write the following radicals in exponential form with positive exponents.

(a) $\left(\sqrt[3]{25}\right)^2$

$= \left(25^{\frac{1}{3}}\right)^2$

$= 25^{\frac{2}{3}}$

(b) $\sqrt[3]{\left(\frac{3}{8}\right)^4}$

$\left(\left(\frac{3}{8}\right)^4\right)^{\frac{1}{3}}$

$= \left(\frac{3}{8}\right)^{\frac{4}{3}}$

(c) $\left(\sqrt{3k}\right)^{-5}$

$= \dfrac{1}{\left((3h)^{\frac{1}{2}}\right)^5}$

$= \dfrac{1}{(3h)^{\frac{5}{2}}}$

Solving Problems Using Exponent Laws

Example:

Biologists use the formula $b = 0.01m^{\frac{2}{3}}$ to estimate the brain mass, *b* kilograms, of a mammal with body mass *m* kilograms. Estimate the brain mass of each animal.

(a) A husky with a body mass of 27 kg

$b = 0.01m^{\frac{2}{3}}$ → m= 27

$$b = 0.01(27)^{\frac{2}{3}}$$
$$b = 0.01\left(\sqrt[3]{27}\right)^2$$
$$b = 0.01(3)^2$$
$$b = 0.01(9)$$
$$b = 0.09kg$$

(b) A polar bear with a body mass of 216 kg.

$b = 0.01m^{\frac{2}{3}}$ → m= 216

$$b = 0.01(216)^{\frac{2}{3}}$$
$$b = 0.01\left(\sqrt[3]{216}\right)^2$$
$$b = 0.01(6)^2$$
$$b = 0.01(36)$$
$$b = 0.36kg$$

Check your Understanding:

1. Simplify the following. Leave your answer with positive exponents.

(a) $(x^3 y^{-\frac{5}{2}})(x^{-1} y^{\frac{1}{2}})$

(b) $\left(\dfrac{50x^2y^4}{2x^4y^7}\right)^{-\frac{1}{2}}$

2. Change the following radicals to exponential form.

(a) $\left(\sqrt{\dfrac{5}{4}}\right)^3$

(b) $\left(\sqrt[3]{-10x}\right)^2$

3. Express each as a single power with a single positive exponent.

(a) $\sqrt{x} \bullet x^3$

(b) $\dfrac{\left(\sqrt[3]{m}\right)^4}{m}$

<u>Concept 7: Number Systems</u>

<u>Vocabulary & Key Concepts</u>

Real Numbers

- the set of numbers that can be represented as fractions and plotted on a number line.
- Includes all rational and irrational numbers

Rational Numbers (*Q*)

- Numbers that are expressed in the form $\frac{m}{n}$, where *m* and *n* are integers, $n \neq 0$
- When written as a decimal they will terminate or repeat
- The Rational number set includes:
    - Integers, *I* (…-2, -1, 0, 1, 2 …)
    - Whole Numbers, *W* (0, 1, 2, 3…)
    - Natural Numbers, *N* (1, 2, 3…)

Irrational Numbers ($\overline{Q}$)

- A number that cannot be expressed in the form $\frac{m}{n}$ where m and n are integers and $n \neq 0$.
- The decimal representation of an irrational number does not terminate or repeat.

<u>Skills</u>

Classifying Numbers

When asked to classify numbers into number sets – start with the 'smallest' group – Natural numbers – then work your way through more inclusive sets.

Example #1

Classify the following numbers into the correct number system(s).

(a) $\sqrt{2}$
Irrational
Real

(b) 3
Natural, Whole,
Integer, Rational, Real

(c) 1.7171171117…
Irrational
Real

(d) $\frac{3}{5}$
Rational
Real

(e) $\sqrt{4}$
Natural, Whole Number
Integer, Rational, Real

(f) $\sqrt{8}$
Irrational
Real

(g) 0
Whole Number
Integer, Rational, Real

(h) π
Irrational
Real

Order Numbers

Use a number line and estimation to order these numbers from least to greatest.

$\sqrt[3]{13}, \sqrt{18}, 9^{\frac{1}{2}}, \sqrt[4]{27}, (-5)^{\frac{1}{3}}$

Perfect Squares

1, 4, 9, 16, 25, 36.

Perfect Cubes

1, 8, 27, 64, 125...

Perfect 4th root

1, 16, 81

$\sqrt[3]{13} \rightarrow$ between $\sqrt[3]{8}$ and $\sqrt[3]{27}$

$\therefore$ 2 and 3

$\sqrt[3]{13} \approx 2.3$

$\sqrt{18} \rightarrow$ between $\sqrt{16}$ and $\sqrt{25}$

$\therefore$ 4 and 5

$\sqrt{18} \approx 4.2$

$9^{1/2} = \sqrt{9}$

$\sqrt[4]{27} \rightarrow$ between $\sqrt[4]{16}$ and $\sqrt[4]{81}$

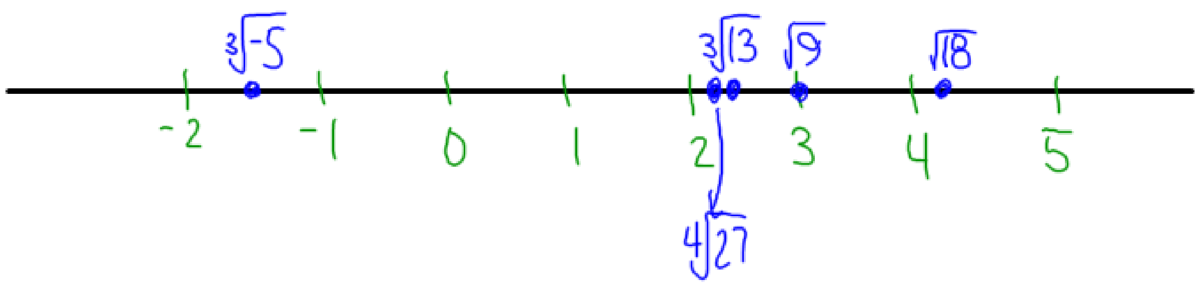
$\therefore$ 2 and 3

$\sqrt[4]{27} \approx 2.2$

$\sqrt[3]{-5} \rightarrow$ between $\sqrt[3]{-1}$ and $\sqrt[3]{-8}$

$\therefore$ between -1 and -2

$\sqrt[3]{-5} \approx -1.5$



Check your Understanding:

1. Name the sets of numbers to which each number belongs.

(a) $\sqrt{12}$ (b) $\sqrt[4]{16}$ (c) $\sqrt{\frac{4}{9}}$ (d) 1.25

2. Use a number line and estimation to order these numbers from least to greatest.

$\sqrt{2}, (-2)^{\frac{1}{3}}, \sqrt[3]{6}, 30^{\frac{1}{4}}, 2\sqrt{5}$